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COVER SHEET FOR TECHNICAL MEMORANDUM

TITLE- Effect of Lunar Slopes on LM Guidance Perturbations Prior to High Gate

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AUTHOR(S)- P. Gunther

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ABSTRACT

Simple analytic expressions are derived for LM guidance perturbations due to initial state errors and to lunar slopes. The main assumptions of the analysis are: no radar noise, uniform radar weighting, constant slope, and restriction to that portion of the trajectory between throttle down and high gate. The results may provide a useful adjunct to the detailed simulations commonly employed.

In the absence of a priori terrain corrections, it is shown that a slope corresponds approximately to a misalignment of axes which leads to extremely large perturbations. These may be further magnified under special conditions, e.g., when the inertial estimate of altitude is more accurate than the radar estimate, or when the slope terminates shortly before high gate. It is believed that appropriate adjustments in the iterative calculation of time-to-go may improve the guidance response characteristics.

A relatively recent guidance modification inputs the terrain profile expected to be encountered. Although the effectiveness of this modification has not been explicitly studied here, the general techniques developed can undoubtedly be applied to estimate the effect of deviations between the a priori and the actual terrain. Of course, when considering contingency landings to a completely different region, the analysis is directly applicable.

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FROM: P. Gunther

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TECHNICAL MEMORANDUM

I. INTRODUCTION

Simulation studies of LM descent guidance performed by Bellcomm,[1] MIT,[2] and MSC[3] have exhibited large pitch perturbations arising from variations in the lunar terrain. Under certain conditions, degraded performance of the landing radar has resulted, so that it has been necessary to constrain the terrain profile encountered. The problem may become even more severe for future lunar missions where landing sites may be located in relatively rough terrain.

This memorandum studies the effect of lunar slopes on the guidance system. The analysis is deterministic--radar noise is neglected in order to concentrate on the relative weighting attached to the radar observations (i.e., the degree of filtering). Only that portion of LM descent between throttle-down and high gate has been treated. After throttle-down lunar curvature is small, and a flat moon approximation can be used. Even more important, the guidance differential equations are linear. The effect of terrain, after filtering of the radar measurements, can then be described in terms of a forcing function and the perturbation readily determined. It turns out that the analysis is largely independent of the nominal trajectory, and the behavior of the system can be explained in relatively simple terms.

The preliminary section 2 summarizes the main features of the LM guidance, including radar updating. Also defined is the nominal trajectory used to illustrate numerically the effect of the perturbations upon the pitch profile. Section 3 analyzes the effect of initial IMU altitude errors. Apart from its intrinsic interest, this analysis provides a useful vehicle for introducing various concepts that enter into the treatment of slopes. Section 4 analyzes similarly the effect of initial velocity errors. In section 5 the response to sloping terrain is studied, both when the slope continues all the way to high gate, and also when only a portion of the terrain slopes. Equations applicable to an arbitrary sequence of slopes, which can approximate most types of terrain, are presented in the appendices.

Some implications of the analysis are discussed in section 6, with comments on the likely gain from employing such techniques as on-board estimation of slope or improved optimization of time-to-go.

II. SUMMARY OF LM DESCENT GUIDANCE*

2.1 Equations

The LM descent employs an iterative quadratic guidance scheme which under ideal conditions converges to desired aim points in position, velocity, and acceleration. Since the equations for the down range component z are identical in form with the equations for the altitude component x, only the latter need be described. Specifically, the inertial acceleration for x is given by

$$\ddot{x}(t) = \ddot{x}_D + c_{1x}(t_f-t) + c_{2x}(t_f-t)^2 \tag{1}$$

t_f is the terminal time for high gate, and is assumed to be constant. $\ddot{x}_D$ is the desired terminal acceleration, i.e., $\ddot{x}(t_f) = \ddot{x}_D$. Integrating (1) gives the velocity equation

$$\dot{x}(t) = \dot{x}_D - \ddot{x}_D(t_f-t) - \frac{1}{2} c_{1x}(t_f-t)^2 - \frac{1}{3} c_{2x}(t_f-t)^3 \tag{2}$$

where $\dot{x}_D$ is desired terminal velocity, i.e., $\dot{x}(t_f) = \dot{x}_D$. Integrating (2) gives position

$$x(t) = x_D - \dot{x}_D(t_f-t) + \frac{1}{2} \ddot{x}_D(t_f-t)^2 + \frac{1}{6} c_{1x}(t_f-t)^3 + \frac{1}{12} c_{2x}(t_f-t)^4 \tag{3}$$

where x_D is desired terminal altitude, i.e., $x(t_f) = x_D$. Solving (2) and (3) for c_{1x} and c_{2x}, for arbitrary time t, gives

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*More complete details are given in References 2 and 4.

$$c_{1x} = \frac{6}{t_f - t}\left[-x_D + \frac{\dot{x}(t) + 3\dot{x}_D}{t_f - t} + 4\,\frac{x(t) - x_D}{(t_f - t)^2}\right] \tag{4}$$

$$c_{2x} = \frac{12}{(t_f - t)^2}\left[\frac{1}{2}\,\ddot{x}_D - \frac{\dot{x}(t) + 2\dot{x}_D}{t_f - t} - 3\,\frac{x(t) - x_D}{(t_f - t)^2}\right] \tag{5}$$

Substituting into (1) gives the guidance differential equation

$$\ddot{x}(t) = \ddot{x}_D - \frac{6}{t_f - t}\,(\dot{x}(t) + \dot{x}_D) - \frac{12}{(t_f - t)^2}\,(x(t) - x_D) \tag{6}$$

Conversely, the solution of (6) is given by (1)-(5). In the absence of errors, the coefficients c_{1x} and c_{2x} in (4) and (5) do not depend on the choice of t. When errors are present, it is reasonable then to use (6) as an iterative guidance law where at any time t the correct best estimates of altitude and velocity $\hat{x}(t)$ and $\hat{\dot{x}}(t)$, are employed.

Several amplifying remarks concerning the above guidance scheme are appropriate; these will serve also to point out some of the simplifying assumptions made in the analysis.

1. Prior to throttle-down, t_f (or $t_{go}=t_f-t$) is computed iteratively so as to keep c_{1z} (so-called down-range jerk) constant. Throttle control is initiated when the required thrust, according to equation (6), corresponds to 52% of maximum thrust available. In order to retain the capability for thrust control, the throttle must range between 10% and 60%. After throttle-down t_f is essentially constant (the down-range component z is not affected by terrain variations) and the coupling between the x and z guidance equations is therefore removed.

2. In order to achieve the inertial acceleration prescribed by (6), the commanded thrust accelerations, $\ddot{x}_c(t)$ and $\ddot{z}_c(t)$, must offset the gravitational acceleration g. Between throttle-down and

and high gate, the down range distance traverses a lunar central angle of about 2°. Hence the assumption of a flat moon with constant g in the x direction affords a satisfactory approximation and

$$\begin{aligned} \ddot{x}_c(t) &= \ddot{x}(t) + g \\ \ddot{z}_c(t) &= \ddot{z}(t) \end{aligned} \tag{7}$$

3. Although the solution of (6) approaches a well-defined limit as $t_{go} \to 0$, the computation becomes highly unstable in the presence of errors. For this reason, as well as others of even more consequence, the latest modification to the LM guidance terminates the braking phase when $t_{go} = 40$ seconds. Nonetheless, the response characteristics of the system can be better understood by supposing that equation (6) extends all the way to $t_{go} = 0$.

4. Radar range measurements are processed every 2 seconds and the altitude is updated by taking a weighted average of the IMU and the radar estimates. A linear radar weighting function is employed which starts at 25,000 feet altitude and reaches its maximum value of .55 at touchdown. A Doppler radar measures three components of velocity, each component being updated cyclically every 6 seconds. The weighting functions actually employed are linear; however, for reasons of mathematical simplicity, only constant weighting functions have been studied here--this departure is not considered important. Two significant operating limitations on the direction of the landing radar beams should be noted. If the incidence angle is too low, the signal-to-noise level may become unacceptably low. At the other extreme, when normal incidence is approached, zero Doppler boundary may be reached.

5. When the estimated altitude, $\hat{x}(t) = x(t) + \hat{b}(t)$, is used in the guidance equation (6), the bias $\hat{b}(t)$ acts as a forcing function to perturb the trajectory. Velocity bias $\hat{c}(t)$ leads to analogous perturbations. The bias functions generated by the (constant) radar filters, as a result of initial altitude errors, initial velocity errors, and terrain slopes, and their corresponding effects upon the guidance, are analyzed in detail in Appendices A and B. A summary discussion, including numerical results, is presented in sections 3 - 5.

2.2 Nominal Trajectory

As mentioned in the introductory section, the perturbation analysis is largely independent of the particular trajectory selected. However, in order to assess more readily the significance of these perturbations, it is useful to present the numerical results in the context of a nominal trajectory. The trajectory used in this study represents a flat moon fit to a trajectory obtained from the Bellcomm descent simulation program, with IMU and radar errors set to zero.* High-gate aim points, initial conditions at the start of throttle-down (t=0), and the guidance parameters derived therefrom (the c's in equation (1)), are tabulated below (units are feet and seconds).

$$x_D = 9866,\ \dot{x}_D = -150.9,\ \ddot{x}_D = -1.73,\ c_{1x} = .04667,\ c_{2x} = -4.17 \times 10^{-4}$$

$$x_0 = 21{,}752,\ \dot{x}_0 = -39.1$$

$$z_D = -42{,}000,\ \dot{z}_D = 704.7,\ \ddot{z}_D = -9.84,\ c_{1z} = .03,\ c_{2z} = -.878 \times 10^{-4}$$

$$z_0 = -190{,}289,\ \dot{z}_0 = 1720.1$$

Average lunar g was taken to be 5.34 ft/sec^2; the duration of the phase was 120 sec.

The above trajectory differs somewhat from those employed in the Apollo 11 and 12 missions.[5,6] In particular, high gate altitudes (x_D) were only 7129 ft. and 7334 ft., respectively. However, the time duration (t_f) between throttle-down and high gate, one of the most significant parameters in this study, was 118 seconds for both missions.

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*The assistance of Gary Bush, Bellcomm, is gratefully acknowledged.

Figure 1 plots the position and velocity components of the assumed nominal trajectory, using equations (2) and (3). $\dot{z}(t)$ does not depart markedly from linearity, suggesting that the higher order contributions due to c_{1z} and c_{2z} are small. Although t=0 is taken to be throttle-down time, altitude updating has actually been going on for some 30 seconds prior. As noted previously, this time period is ignored in the analysis.

Figure 2 plots the components of acceleration, (equations (1) and (7)) and also the total given by

$$a_c(t) = [\ddot{x}_c^{\,2}(t) + \ddot{z}^2(t)]^{1/2} \tag{8}$$

Also shown is the throttle setting T(t) defined by

$$T(t) = \frac{F(t)}{F_{max}}$$

$$= \frac{M_o}{F_{max}} a_c(t) e^{-\frac{1}{V_E}\int_0^t a_c(\tau)d\tau} \tag{9}$$

where

F_{max} = maximum engine thrust = 10,500 g_E lbs. force*

$F(t)$ = commanded thrust at time t

M_o = LM mass at throttle-down = 21,530 lbs. mass

V_E = exhaust velocity = 9813 fps

For the nominal trajectory, the throttle is initially (by definition) 52%, and increases gradually, remaining just below 60%. The values of $\ddot{z}$ are sufficiently large that perturbations in $\ddot{x}$ are unlikely to cause T to fall below 10%. For T to exceed 60% near high gate, a positive perturbation of 2.45 ft/sec^2 is required.

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*In practice, F_{max} increases slightly with time.

Figure 3 shows the nominal pitch profile $\phi(t)$ defined by

$$\phi(t) = \tan^{-1} \frac{\ddot{x}_c(t)}{\ddot{z}(t)}, \quad 0 \leq \phi \leq 90° \tag{10}$$

The pitch at the beginning and at the end of the phase are fairly close, and the total variation is about 10°. Also shown in the figure are the pitch boundaries corresponding to the operational limits on the radar beam incidence angle. (These have been estimated using the boundary curves in Reference 2.) It appears that only toward the early portion of the phase do these constraints have any possibility of being violated.

III. INITIAL ALTITUDE ERROR

Suppose that at t=0 the IMU estimate of altitude is in error by an amount b_0.* This includes not only the propagation of state errors from time of ignition but also any error in estimate of radius of the moon. Altitude radar measurements are processed every Δ(=2) seconds, with the filter using a weighted average of the IMU estimate and the radar measurement to provide an updated estimate of altitude. Assuming the radar weighting function to be constant, say w, and ignoring random radar errors, then a fraction w of the bias will be removed at each updating time t_n. The remaining bias $\hat{b}_n$ is simply

$$\begin{aligned} \hat{b}_n &= (1-w)\, \hat{b}_{n-1} \\ &= b_0 (1-w)^n \end{aligned} \tag{11}$$

When w is not too large, one can use a continuous approximation to the discrete solution (11), which leads to an exponential fall-off:

$$\hat{b}(t) = b_0 e^{-\alpha t} \tag{12}$$

where

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*The terminology, IMU error or bias, is used to denote the state vector estimation error generated solely from processing of accelerometer and gyro readings, prior to the time when radar data is available.

$$\alpha = \frac{w}{\Delta} \tag{13}$$

α can be interpreted as the correction rate. (Note that w=1 corresponds in the continuous case to $\alpha=\infty$.)

By substituting $\hat{x}(t) = x(t) + \hat{b}(t)$ into the guidance equation (6), the perturbed acceleration, $\delta\ddot{x}(t)$, due to $\hat{b}(t)$ is seen to be given by the response of (6) to the forcing function $-12\hat{b}(t)/(t_f-t)^2$. A formal analysis is given in Appendix B--more intuitive derivations are presented below.

First note that the case when w=1, whereby the entire bias is removed at t=0, is equivalent to a change in initial conditions from x_0 to x_0-b_0 ($b_0>0$ means that the prior estimate was high). The changes in the response coefficients, c_{1x} and c_{2x}, of equation (1) are readily obtained from equations (4) and (5) with t=0, giving $\delta c_{1x} = -24b_0/t_f^{\,3}$, $\delta c_{2x} = 36b_0/t_f^{\,4}$. Hence the perturbation in (1) is simply the quadratic

$$\delta\ddot{x}(t) = \frac{12b_0}{t_f^{\,2}}\left[\frac{3}{t_f^{\,2}}(t_f-t)^2 - \frac{2}{t_f}(t_f-t)\right]$$

$$= \frac{12b_0}{t_f^{\,2}}\left(1-\frac{t}{t_f}\right)\left(1-\frac{3t}{t_f}\right) \tag{14}$$

Note that if altitude filtering had started at time t_1, and the entire bias b_0 had been removed then, the response would be the same as (14) except that t_f is replaced by $t_{f_1} \equiv t_f - t_1$ and t by $t-t_1$.

The general case, w<1, with bias $\hat{b}(t)$, can be treated by considering t_1 variable (say s) and noting that in the infinitesimal interval (s, s+Δs) the bias removed is $\hat{b}(s+\Delta s) - \hat{b}(s) \simeq \hat{b}'(s)\Delta s$. At time t (t>s) the contribution to the perturbation is, by (14)

$$\frac{12}{(t_f-s)^2}\left(1-\frac{t-s}{t_f-s}\right)\left(1-3\,\frac{t-s}{t_f-s}\right)\hat{b}'(s)\Delta s$$

Hence the total perturbation at time t is given by the integral over all $s (0 \leq s \leq t)$, namely

$$\delta\ddot{x}(t) = 12 \int_0^t \frac{\hat{b}'(s)}{(t_f - s)^2} \left(1 - \frac{t-s}{t_f - s}\right)\left(1 - 3\,\frac{t-s}{t_f-s}\right) ds \qquad (15)$$

(15) does not depend upon the nominal trajectory. When $\hat{b}(t)$ is given by (12), equation (15) can be evaluated in terms of the exponential integral, for which tables are available[7].

The results are plotted in Figure 4 for several values of w. Since, as in (14), the response is proportional to $12\ b_0/t_f^{\,2}$, a convenient normalization is obtained by choosing b_0 = 1200 ft.--with t_f = 120 sec., then $12\ b_0/t_f^{\,2} = 1$. Note also that the response depends only upon non-dimensional time t/t_f [or $(t-t_1)/(t_f-t_1)$] and upon the total "equivalent" number of corrections αt_f [or $w(t_f-t_1)/\Delta$].

The perturbations in Figure 4 are typically "oscillatory" with a positive peak in the beginning portion, and in the later portion what might suggestively be referred to as a "restoring" peak. The smaller w is, the more gradually the bias is removed, the smaller is the initial peak and the later the occurrence of the restoring peak. When w approaches zero the perturbation is identically zero, but the terminal bias is $-b_0$. In general the terminal biases in altitude, (vertical) velocity and acceleration are given by:

$$\begin{aligned} \delta x(t_f) &= -b_0 e^{-\alpha t_f} \\ \delta\dot{x}(t_f) &= 2b_0\alpha e^{-\alpha t_f} \\ \delta\ddot{x}(t_f) &= -6b_0\alpha^2 e^{-\alpha t_f} \end{aligned} \qquad (16)$$

Using the nominal values of t_f = 120 sec., b_0 = 1200 ft., and Δ = 2 sec., the terminal biases for the w's in Figure 4 are:*

w	0	.05	.1	.2	1.0
$\delta x(t_f) \sim$ ft	-1200	-60	-3	-.007	0
$\delta \dot{x}(t_f) \sim$ fps	0	3	.3	.0014	0
$\delta \ddot{x}(t_f) \sim$ ft/sec^2	0	-.225	-.045	-.0004	0

For w=.1 the terminal errors are negligible, and even for w=.05 they are relatively small.

Using the nominal trajectory described in section 2.2, the resultant pitch and total acceleration have been determined. The pitch profile in Figure 4P assumes $b_0 = \pm$ 2400 ft. The curve for w=0 represents the nominal, unperturbed pitch (Figure 3), and w=1 is the limiting solution. From examination of the radar operational limits in Figure 3, it appears that only for negative biases and only during the early portion of the phase are the pitch perturbations likely to be operationally significant; hence in this region the use of low weighting for the radar appears to be most advantageous.

Regarding total acceleration (Figure 5), the most significant effects occur at the later times where the possibility exists of exceeding the 60% throttle limit. Again this problem arises only for negative bias, and the weighting function appears to be relatively unimportant.

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*The actual biases, corresponding to t_{go} = 40 seconds, would be somewhat different (see Appendix B).

IV. INITIAL VELOCITY ERROR

Both (vertical) velocity filtering and altitude filtering affect the perturbation resulting from an initial velocity error c_0. The relevant parameters are: (i) Altitude radar weight w, with correction rate $\alpha = w/\Delta$; (ii) Velocity radar weight v, with correction rate $\beta = v/\Delta_v$ (updating interval $\Delta_v = 6$ sec); and (iii) Time t_1 that velocity radar updating begins. The analysis requires the determination of both the velocity bias function $\hat{c}(t)$, which is unaffected by altitude filtering, and also the induced altitude bias $\hat{b}_v(t)$.

The velocity bias is analogous to that for altitude bias discussed in the preceding section. Specifically,

$$\hat{c}(t) = \begin{cases} c_0 & t < t_1 \\ c_0 e^{-\beta(t-t_1)} , & t \geq t_1 \end{cases} \tag{17}$$

In the special case where $t_1 = 0$ and the entire bias c_0 is removed at t=0, i.e., v=1 ($\beta = \infty$), the perturbation is equivalent to a change in initial velocity from $\dot{x}_0$ to $\dot{x}_0 - c_0$. From equations (1), (4), and (5) one gets, after some algebra,

$$\delta\ddot{x}(t) = \frac{6c_0}{t_f}\left(1 - \frac{t}{t_f}\right)\left(1 - \frac{2t}{t_f}\right) \tag{18}$$

The response is proportional to c_0/t_f and depends upon non-dimensional time t/t_f. Equation (18) holds for any altitude filter weight w, since when v=1 no altitude bias is induced.

Before generalizing (18) to v<1, it is convenient to discuss first the special cases of velocity filtering only and altitude filtering only.

4.1 Velocity Filtering Only (w=0)

In the absence of altitude filtering, the previous result (18) for v=1 holds when the starting time is $t_1 > 0$ except that t_f is replaced by $t_{f_1} = t_f - t_1$ and t by $t - t_1$. By the same argument leading to (15), when v<1 the overall perturbation is obtained by integrating (18) for variable t_1 (=s, say) and with c_0 replaced by $\hat{c}'(s)\Delta s$. Specifically,

$$\delta\ddot{x}(t) = 6 \int_0^t (1 - \frac{t-s}{t_f-s})(1 - 2\frac{t-s}{t_f-s})\ \hat{c}'(s)ds$$

where $\hat{c}'(s)$ is obtained from differentiating (17). The resulting perturbations are plotted for various v in Figure 6 assuming t_1=0 and c_0=20 fps, which gives the normalization $6\ c_0/t_f=1$. For $t_1>0$, only a change in scale is required. Qualitatively the features are similar to those for initial altitude error (Figure 4) except that the "restoring" peak is somewhat smaller. The corresponding pitch profiles are plotted in Figure 6P, assuming $c_0 = \pm\ 13\ 1/3$ fps.

Although knowledge of the altitude bias function was not needed in deriving the acceleration perturbation, its value is of interest. Since w=0

$$\hat{b}_v(t) = \int_0^t \hat{c}(t)dt$$

Integrating (17) gives

$$\hat{b}_v(t) = \begin{cases} c_0 t & ,\quad t < t_1 \\ c_0 t_1 + \frac{c_0}{\beta}(1 - e^{-\beta(t-t_1)}), & t \geq t_1 \end{cases}$$

After t_1, the bias builds up asymptotically only by the additional amount c_0/β.

4.2 Altitude Filtering Only (v=0)

Consider first the case where the altitude radar weight w=1. In the time interval between s and s+Δs the amount of altitude bias removed is $c_o \Delta s$. The resulting perturbation at any later time t>s is, from equation (14),

$$12 \frac{c_0 \Delta s}{(t_f - s)^2} \left(1 - \frac{t-s}{t_f - s}\right)\left(1 - 3 \frac{t-s}{t_f - s}\right)$$

The total perturbation at time t is the sum of the perturbations arising from each s between 0 and t, or

$$\delta \ddot{x}(t) = 12 c_0 \int_0^t \frac{1}{(t_f - s)^2} \left(1 - \frac{t-s}{t_f - s}\right)\left(1 - 3 \frac{t-s}{t_f - s}\right) ds$$

$$= 12 \frac{c_0}{t_f} \left(\frac{t}{t_f}\right)\left(1 - \frac{t}{t_f}\right) \tag{19}$$

When w<1, in each interval (s, s+Δs) the bias is increased by an amount $(1-w)\, c_0 \Delta s$. Subsequent altitude filtering decreases this, at time t>s, by the factor $e^{-\alpha(t-s)}$. Hence the overall bias is

$$\hat{b}_v(t) = (1-w) c_0 \int_0^t e^{-\alpha(t-s)} ds$$

$$= (1-w) \frac{c_0}{\alpha} \left(1 - e^{-\alpha t}\right) \tag{20}$$

An exponential buildup results, reaching a "steady state" bias of $(1-w)\, c_0/\alpha$. The perturbation effect of (20) is the same as for an initial altitude error, of negative magnitude, given by equation (15) in section 3, which effect is subtracted from the perturbation (19). It is interesting to note that the responses (Figure 7) do not contain a negative "restoring" peak. The pitch profiles are presented in Figure 7P, assuming c_0 = 13 1/3 fps.

4.3 Both Velocity and Altitude Filtering

We first consider the case when w=1. The perturbation due to altitude filtering alone (i.e., v=0) was previously given by equation (19); while when v=1 (and t_1=0) the total perturbation was given by (18). Hence the difference can be viewed, for v=1, as the additional perturbation due to velocity filtering, namely

$$\frac{6c_0}{t_f}\left(1 - \frac{t}{t_f}\right)\left(1 - 4\,\frac{t}{t_f}\right) \tag{21}$$

More generally when v < 1 the additional perturbation due to velocity filtering is obtained from (21) in the usual way, giving

$$6\int_0^t \frac{1}{t_f-s}\left(1 - \frac{t-s}{t_f-s}\right)\left(1 - 4\,\frac{t-s}{t_f-s}\right)\hat{c}'(s)\,ds \tag{22}$$

The resulting perturbations for w=1, i.e., the sum of (19) and (22), are plotted in Figure 8 for t_1=0. Except when v=0 the results are qualitatively similar to w=0 (Figures 6 and 6P). A "restoring" peak is present which increases in magnitude as v decreases (but eventually decreases for v sufficiently small).

The analysis of the general case, with arbitrary w, v, and t_1, is essentially a combination of the results of the previous special cases. Details are presented in Appendices A and B. Figure 9 shows the perturbations for w=.1 and for t_1=0 and 40 sec. Figure 10 similarly shows the results for v=.2. When t_1=40 sec, the perturbations become quite large since the velocity filtering contribution is now accomplished within the smaller duration $t_{f_1} = t_f - t_f$. Pitch profiles are shown in Figure 9P.

Terminal biases in altitude, velocity, and acceleration, for t_1=0 and 40 seconds, c_0=20 fps, and t_f=120 seconds, are presented in Table 1.

V. TERRAIN SLOPES

5.1 Constant Slope with Complete Filtering (w=1)

Suppose that the terrain has constant slope θ throughout the entire trajectory between throttle-down and high gate. Assume also that the (range) radar measures without error the vertical height above the surface.* The terrain bias, $\tilde{b}(t)$, referenced to the altitude at high gate, is then

$$\tilde{b}(t) = \tan\theta(z_D - z(t)) \quad (23)$$

When w=1, as is initially assumed here, the response of the guidance to $\tilde{b}(t)$ reflects the effect of terrain, and no filtering component is present.

Consider first the simple case where down range velocity is constant, i.e.,

$$z(t) = z_D - \dot{z}_D(t_f - t) \quad (24)$$

(23) then becomes

$$\tilde{b}(t) = \dot{z}_D \tan\theta(t_f - t) \quad (25)$$

Since (25) is linear in t, the resulting perturbation, say $\delta_0\ddot{x}(t)$,† is equivalent to one arising from a constant velocity error of amount $\dot{z}_D \tan\theta$ (with altitude filtering only) so that from equation (19)

$$\delta_0\ddot{x}(t) = 12\tan\theta \frac{\dot{z}_D}{t_f}\left[\frac{t}{t_f}\left(1 - \frac{t}{t_f}\right)\right] \quad (26)$$

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*Actually, for sloping terrain a slight bias, whose magnitude depends upon the pitch, arises from the fact that the range measurement must be projected onto the local vertical.

†In Appendix B, equation (83), the proportionality constant is not included. Likewise for δ_1, δ_2, δ_3 below.

For an intuitive explanation, note that if the vehicle were moving at uniform horizontal velocity $\dot{z}_D$, with $\dot{x}(t)=0$, the terrain would appear to be rising (for θ positive) at a rate $\dot{z}_D \tan\theta$, which thus represents the relative vertical velocity. This implies further that the bias could be removed, or at least made constant, by rotating the x-z axes through an angle θ so that the new x-axis is parallel to the terrain.

It is of some interest that (26) can also be derived by substituting (25) into the guidance differential equation (6), and observing that the aim points, x_D and $\dot{x}_D$, are effectively changed (relative to the altitude at t=0) by the amounts $\dot{z}_D t_f \tan\theta$ and $2\dot{z}_D \tan\theta$, respectively.

The actual nominal down-range trajectory is not linear, but rather a quartic (equation (3))

$$z(t) = z_D - \dot{z}_D(t_f-t) + \frac{\ddot{z}_D}{2}(t_f-t)^2 + \frac{c_{1z}}{6}(t_f-t)^3 + \frac{c_{2z}}{12}(t_f-t)^4 \qquad (27)$$

The bias (23) now corresponds to that of a time-varying velocity error. The contributions of the higher order terms in (27) can be determined using the same method as in equations (15) and (19), namely

$$\begin{aligned}\delta\ddot{x}(t) &= 12\int_0^t \left[\frac{1}{(t_f-s)^2}\left(1-\frac{t-s}{t_f-s}\right)\left(1-3\,\frac{t-s}{t_f-s}\right)\right]\tilde{b}'(s)ds \\ &= 12\tan\theta\int_0^t [\qquad]\,\dot{z}(s)ds\end{aligned} \qquad (28)$$

In particular, the contribution of $\ddot{z}_D$, say $\delta_1\ddot{x}(t)$, is†

$$\delta_1\ddot{x}(t) = -12\tan\theta\,\ddot{z}_D\cdot\left[\frac{t}{t_f}\left(1-\frac{3}{2}\frac{t}{t_f}\right)\right] \qquad (29)$$

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† $\ddot{z}_D$ is negative since the vehicle is decelerating. Also $\delta_1\ddot{x}(t)$ can be derived in the same way as for $\dot{z}_D$, by considering the equivalent changes in the aim points.

Whereas in (26) the response to $\dot{z}_D$ is inversely proportional to t_f, in equation (29) the magnitude of δ_1 depends only upon $\ddot{z}_D$.

From (28), the contributions of c_{1z} and c_{2z} are readily found to be

$$\begin{aligned}\delta_2\ddot{x}(t) &= -12\tan\theta\ c_{1z}t_f\left[(1-\frac{t}{t_f})(\frac{3}{2}\frac{t}{t_f} + \ln(1-\frac{t}{t_f})\right]\\ \delta_3\ddot{x}(t) &= 12\tan\theta\ c_{2z}t_f^2\left[(1-\frac{t}{t_f})\left\{\frac{1}{3}\frac{t}{t_f} + (1-\frac{t}{t_f})\ln(1-\frac{t}{t_f})\right\}\right]\end{aligned} \tag{30}$$

Figure 11 plots the $\delta_i\ddot{x}(t)$, i=0, 1, 2, 3, using nominal values for the coefficients and (for convenience of scaling) $\theta=2°$. The contributions of c_{1z} and c_{2z} are quite small, so that the overall perturbation depends on the nominal trajectory only through the aim points $\dot{z}_D$ and $\ddot{z}_D$ (and t_f). Initially the responses to each of these are approximately equal, but near high gate the large negative response from $\ddot{z}_D$ dominates.

5.2 Effect of Altitude Filtering (w<1)

Let the altitude weighting be w, with correction rate $\alpha=w/\Delta$. Assuming that the radar and the IMU agree just prior to occurrence of the slope (at t=0), the filtered bias is

$$\hat{b}_T(t) = e^{-\alpha t}\left[\tilde{b}(0) + \int_0^t \alpha e^{\alpha s}\ \tilde{b}(s)ds\right] \tag{31}$$

Integrating by parts gives

$$\hat{b}_T(t) = \tilde{b}(t) + b_w(t) \tag{32}$$

where

$$b_w(t) = -e^{-\alpha t}\int_0^t e^{\alpha s}\ \tilde{b}'(s)ds \tag{33}$$

$b_w(t)$ represents the filter component of the bias. For w=1, $\lim_{\alpha\to\infty} b_w(t) = 0$; while for w=0, $\lim_{\alpha\to 0} b_w(t) = \tilde{b}(0) - \tilde{b}(t)$ and $\tilde{b}_T(t) \to \tilde{b}(0)$.

When down range velocity is constant, $\tilde{b}(t)$ is given by (25) and $b_w(t)$ becomes

$$b_w(t) = \tan\theta \, \frac{\dot{z}_D}{\alpha} \, (1 - e^{-\alpha t}) \tag{34}$$

The discrete version of the filter bias (34) is of some interest. The incremental bias from successive radar observations, due to terrain slope, is $\dot{z}_D \tan\theta \cdot \Delta = \partial\tilde{b}$ say. Since the filter retains the bias $(1-w)\partial\tilde{b}$ from the preceding update, the cumulative bias at the n^{th} update is*

$$\delta\tilde{b}[1 + (1-w)+\ldots+(1-w)^{n-1}] = \partial\tilde{b} \, \frac{1-(1-w)^n}{w} \; , \; 0<w\leq 1 \tag{35}$$

Letting $n=t/\Delta$ and then $\Delta\to 0$ gives (34). For variable $\partial\tilde{b}$, say $\partial\tilde{b}_j$, (35) becomes $\sum_{j=1}^{n} \partial\tilde{b}_j (1-w)^{n-j}$ which approaches (33).

The filter bias (34) can be further subdivided into the "steady state" component, $\tan\theta \cdot \dot{z}_D/\alpha$, and the "transient", $-e^{-\alpha t} \tan\theta \cdot \dot{z}_D/\alpha$, the negative sign indicating a buildup. The filter transient yields the same perturbation as an initial altitude error (compare equations (20) and (34)), which effect is then subtracted from the terrain perturbation δ_0. The steady state bias, being constant, has no effect.

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*Equation (35) assumes that each radar observation is taken at the beginning of the Δ interval. If the observation occurs at the end, (35) is multiplied by 1-w.

In the general case of nonlinear z(t), the filter bias can be expressed in terms of a derivative operator D (see Appendix A, equations (34)-(36)) as follows.

$$b_w(t) = D\tilde{b}(t) - D\tilde{b}(0) \cdot e^{-\alpha t} \tag{36}$$

The first term represents the steady state component

$$\begin{aligned} D\tilde{b}(t) &= \frac{-b'(t)}{\alpha} + \frac{b''(t)}{\alpha^2} - \frac{b'''(t)}{\alpha^3} + \frac{b^{(iv)}(t)}{\alpha^4} \\ &= \tan\theta \left[\frac{\dot{z}_D}{\alpha} - \frac{\ddot{z}_D}{\alpha}(t_f - t) - \frac{\ddot{z}_D}{\alpha^2} + \text{terms in } c_{1z} \text{ and } c_{2z} \right] \end{aligned} \tag{37}$$

while the magnitude of the transient is

$$D\tilde{b}(0) = \tan\theta \; \frac{\dot{z}_D}{\alpha} - \frac{\ddot{z}_D t_f}{\alpha} - \frac{\ddot{z}_D}{\alpha^2} + \ldots \tag{38}$$

The perturbation due to $b_w(t)$ is readily determined: the effect of the transient $D\tilde{b}(0)$ is the same as before (i.e., an initial altitude error), while the effect of $D\tilde{b}(t)$, namely $\ddot{z}_D \tan\theta/\alpha \cdot (t_f - t)$ and higher order terms, can be obtained from formulas (29) and (30) in the previous section.

Figure 12 shows the overall perturbations for various w. (The difference between the w=1 curve represents the response to $b_w(t)$.) Near high gate the perturbations are quite large--as noted previously this effect is due to $\ddot{z}_D$. The pitch profiles are plotted in Figure 12P. Terminal biases are shown in Table 2.

5.3 Variations on Constant Slope

1. When the slope begins at time $t_1 > 0$, the preceding discussion can be applied merely by translating the time origin to t_1. Since $\delta_0\ddot{x}(t)$, the contribution from $\dot{z}_D$, is inversely

proportional to t_f, this effect increases; but the contribution from $\ddot{z}_D$ is unchanged. The overall perturbation is illustrated in Figures 13 and 13P when t_1=60 sec.

2. The previous results assumed initial agreement between the IMU and the radar. If, however, the IMU estimate were actually exact (relative to the altitude at high gate), then the perturbations become more severe since the guidance system eventually adopts the radar values. Omitting the constant $\tilde{b}(0)$ in equation (31) is equivalent to subtracting the response to an initial altitude error, so that the overall perturbation is given by the difference between the results in Figures 4 and 12. (The appropriate value of b_0 to use with Figure 4 is $\tan\theta \cdot (z_D - z_0)$ = 5178 ft. for θ=2° and t_f=120 sec.--the response is plotted in Figure 15, which is discussed under 3 below.) The net result is that there is a large initial, as well as terminal, perturbation as shown in Figures 14 and 14P. If the 2° slope were assumed to continue from high gate to the site, and if the initial IMU altitude error were zero when referenced to the altitude at the site, the perturbations are even more pronounced. This is shown for w=.1 by the dashed curve in Figure 14.

3. Suppose the radar data were used merely to estimate the slope of the terrain, and by extrapolation an estimate then obtained of the altitude at high gate. If, relative to this altitude, the initial IMU bias were zero then clearly no correction to the trajectory would be needed. Otherwise, the bias would be filtered in the same way as for an initial altitude error. For the case where the IMU and radar initially agree, it turns out that the perturbations (Figures 15 and 15P) are appreciably larger than when the data are filtered in the usual terrain following manner with the bias being removed more gradually (Figures 12 and 12P). In the former instance, however, the terminal velocity and acceleration biases are reduced (see Table 2).

4. Suppose that the slope θ terminates at some time t_1 (prior to high gate) and that the terrain is flat thereafter. At t_1 the radar sees a change in slope of amount $-\theta$ which induces a perturbation similar to that at t=0 (section 5.2) but with $t_f - t_1$ in place of t_f and $t - t_1$ in place of t. For $t > t_1$ the overall perturbation is the sum of the two separate perturbations. The results are plotted in Figures 16 and 16P for various slope durations t_1, and for w=1.0 and .1. The "restoring" peak is seen to be quite large, becoming progressively more severe the

closer to high gate that the slope terminates. Somewhat surprisingly, if the slope does not terminate at all (t_1=120) the perturbation becomes smaller, although in this case the terminal errors are larger (Table 2). Similar, but even larger perturbations are obtained if the slope does not begin to appear until time-to-go is 60 sec. (Figures 17 and 17P). The analysis for an arbitrary sequence of slopes is presented in Appendices A and B.

VI. CONCLUSIONS AND COMMENTS

The main conclusion of this study is that sloping terrain presents a potentially serious problem for the LM guidance. Several possible methods for alleviating this problem are briefly discussed below.

The large perturbations are believed to be basically inherent to the present guidance scheme, in particular, the fact that the altitude and down range guidance are uncoupled and consequently time-to-go remains constant when the terrain varies. (Neither of these characteristics is present in the optimal bilinear tangent steering law; this law yields a "minimal path", and hence probably also minimizes pitch perturbations.) Further study would be required to determine whether a simple, effective modification can be found to the present subroutine for calculating time-to-go.

Although it would probably not be too difficult to identify and estimate any lunar slope from the radar data, it is not evident how such knowledge could be usefully exploited. As noted in section 5.3, converting the terrain bias to an (initial) position bias by extrapolating the slope to high gate, leads to even more severe perturbations when the radar and IMU initially agree. This is because the bias is then removed at a much faster rate than when the usual terrain following scheme is employed. A similar effect may arise if the coordinate system were to be rotated whenever a slope change occurs. Another alternative--and the one that has recently been adopted in the LM guidance--is to input a prior estimate of the elevation profile for the expected ground track. However, the sensitivity to deviations in the nominal trajectory may require further study. The analysis in section 5 is expected to be applicable to the difference between the a priori and the actual slope.

Finally, some improvement may be possible by allowing the filter weighting to depend upon the magnitude of slope and the remaining time-to-go. In general, the smaller the radar weighting used, the smaller the perturbation. Although the terminal high gate errors are thereby increased, in most instances these appear to remain quite small.

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Attachments
- References
- Appendices A - B
- Tables 1 - 2
- Figures 1 - 17P
- Figure A1

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APPENDIX A

DERIVATION OF BIAS FORMULAS

A.1 EFFECT OF FILTER ON BIAS

This appendix derives expressions for the altitude bias as affected by filtering of the radar observations starting at the throttle-down point (t=0). It turns out to be convenient, as indicated in Figure A1, to reference down-range z to the nominal landing site, while altitude x is referenced to the terrain at high gate.† Other quantities of interest, illustrated in Figure A1, are:

$x(t)$ = true altitude

$\tilde{x}(t)$ = radar measurement of altitude (assumed to be measured vertically above lunar surface and without error)

$\hat{x}(t)$ = updated estimate of altitude, after filtering, used in the guidance equation

$$\hat{b}(t) \equiv \hat{x}(t)-x(t) = \text{bias in estimated altitude} \tag{1}$$

$$\tilde{b}(t) \equiv \tilde{x}(t)-x(t) = \text{terrain bias in radar altitude measurement} \tag{2}$$

Similar quantities can be defined for velocity $\dot{x}$. In particular,

$$\hat{c}(t) = \hat{\dot{x}}(t) - \dot{x}(t) = \text{bias in estimated velocity} \tag{3}$$

Initially we present a simplified description of the filtering process. Suppose altitude updating is performed every Δ seconds, say at times $t_1, t_2, \ldots, t_n, \ldots,$ where $t_n = n\Delta$. Let $\hat{x}^*$ denote the extrapolated estimate of altitude derived from IMU measurements during the interval Δ. Specifically,

$$\hat{x}^*(t_{n+1}) = \hat{x}(t_n) + \Delta \cdot \hat{\dot{x}}(\bar{t}_n) \tag{4}$$

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†Assuming a flat moon, altitude referred to the site requires only the addition of a fixed constant B (see equation (45)).

where $\hat{\dot{x}}$ is the estimated velocity at the midtime $\bar{t}_n$ between t_n and t_{n+1}.[†] The filter updates the altitude using a weighted average of the extrapolated and radar estimates, given approximately by

$$\hat{x}(t_{n+1}) = (1-w_{n+1})\ \hat{x}^*(t_{n+1}) + w_{n+1}\tilde{x}(t_{n+1}) \tag{5}$$

where w_{n+1} is the weight given the radar measurement at time t_{n+1}.

In order to express the filter equation (5) in terms of biases, we need to determine the bias in the extrapolated estimate (4). The true altitude, for which the extrapolation (4) is an estimate, is given by

$$x(t_{n+1}) = x(t_n) + \Delta \cdot \dot{x}(\bar{t}_n) \tag{6}$$

Subtracting (6) from (4) gives for the extrapolated bias

$$\hat{x}^*(t_{n+1}) - x(t_{n+1}) = \hat{b}(t_n) + \Delta \cdot \hat{c}(t_n) \tag{7}$$

[†] In fact $\hat{\dot{x}}(\bar{t}_n) = \hat{\dot{x}}(t_n) + \frac{1}{2}\, a_n\Delta$, where a_n is the combined thrust and gravitational accelerations, assumed constant during the interval. If a_n is also assumed to be determined without error, it follows that $\hat{c}(\bar{t}_n) = \hat{c}(t_n)$, which fact is used in (7).

Subtracting now $x(t_{n+1})$ from both sides of (5) gives, in view of (7),

$$\hat{b}(t_{n+1}) = (1-w_{n+1})\hat{b}(t_n) + (1-w_{n+1})\ \Delta \cdot \hat{c}(t_n) + w_{n+1}\tilde{b}(t_{n+1}) \qquad (8)$$

In the actual LM guidance computer,[4] updating occurs at the time, $t_n+\delta$ $(0 \le \delta \le \Delta)$, when the radar measurement is taken. The filter equation (5), which corresponds to $\delta=\Delta$, is replaced by

$$\hat{x}(t_{n+1}) = \hat{x}^*(t_{n+1}) + w_{n+1}[\tilde{x}(t_n+\delta) - \hat{x}^{**}(t_n+\delta)] \qquad (5a)$$

where $\hat{x}^{**}$ is a simple linear extrapolation of previous altitudes given by

$$\hat{x}^{**}(t_n+\delta) = \hat{x}(t_n) + \delta \cdot \frac{\hat{x}(t_n) - \hat{x}(t_{n-1})}{\Delta} \qquad (9)$$

If one assumes that

$$\frac{\hat{x}(t_n) - \hat{x}(t_{n-1})}{\Delta} \simeq \hat{\dot{x}}(t_n) + \frac{1}{2} a_n \delta$$

which has only a small second order error, then w_{n+1} in the second term of (8) is replaced by

$$\bar{w}_{n+1} \equiv \frac{\delta}{\Delta} w_{n+1} \qquad (10)$$

and equation (8) becomes

$$\hat{b}(t_{n+1}) = (1-w_{n+1})\hat{b}(t_n) + (1-\bar{w}_{n+1})\ \Delta \cdot \hat{c}(t_n) + w_{n+1}\tilde{b}(t_n+\delta) \qquad (11)$$

Note that the later in the interval that the radar measurement is taken, the larger is δ and the smaller the contribution of velocity bias $\hat{c}(t_n)$. (In the computations, it was assumed that $\delta = \Delta$, i.e., $\bar{w} = w$.)

Since the filter is linear, the total altitude bias $\hat{b}(t)$ can be considered to be the sum of individual effects due to initial altitude error $(\hat{b}_{IC}(t))$, initial velocity error $(\hat{b}_V(t))$, and terrain variations $(\hat{b}_T(t))$. The succeeding sections analyze these separately.

A.2 INITIAL ALTITUDE ERROR $(\hat{b}_{IC}(t))$

When the velocity and the terrain biases are zero, (11) reduces to[†]

$$\hat{b}(t_{n+1}) = (1-w_{n+1})\hat{b}(t_n) \tag{12}$$

Assuming that the radar weighting is assigned beforehand, (i.e., the w_i depend only on t and not on x), (12) is a simple difference equation. At t=0, let the initial IMU bias be $b_0 = \hat{x}_0 - x_0$. The solution of (12) is then

$$\hat{b}(t_n) = b_0 \prod_{1}^{n} (1-w_i) \tag{13}$$

Consider now the simplest case of constant weights, say $w_i = w$. Then

$$\hat{b}(t_n) = b_0(1-w)^n \tag{13a}$$

[†]For simplicity the subscript IC (denoting initial altitude condition) is omitted from $\hat{b}_{IC}$.

Since $n = t_n/\Delta$, we can write

$$\hat{b}(t_n) = b_0 e^{-\alpha_D t_n} \tag{14}$$

where

$$\alpha_D = -\frac{1}{\Delta}\ln(1-w)$$

$$= \frac{w}{\Delta}\left(1+\frac{w}{2}+\frac{w^2}{3}+\ldots\right) \tag{15}$$

The solution (14) suggests consideration of the (continuous) differential equation approximation to the (discrete) difference equation (12). We can rewrite (12) as

$$\frac{\hat{b}(t+\Delta)-\hat{b}(t)}{\Delta} = -\alpha(t)\hat{b}(t) \tag{16}$$

where

$$\alpha(t) = \frac{w(t)}{\Delta} \tag{17}$$

For Δ sufficiently small, the left hand side is approximately $\hat{b}'(t)$, so that approximately

$$\hat{b}'(t) = -\alpha(t)b(t) \tag{18}$$

and hence

$$\hat{b}(t) = b_0 e^{-\int_0^t \alpha(s)ds} \tag{19}$$

When w(t) is constant, say w, and $\alpha = w/\Delta$, then

$$\hat{b}(t) = b_0 e^{-\alpha t} \qquad (20)$$

Comparing this with (14) and (15), one sees that (20) underestimates the bias. A second order approximation to the discrete solution is

$$\hat{b}(t) = b_0 e^{-\alpha(1+\frac{w}{2})t} \qquad (21)$$

For w reasonably small, the first order solution (20) will be sufficiently accurate. (For later use, observe that when w=1 the discrete solution is $\hat{b}(t_n) = 0$ $(n>0)$, which corresponds in the continuous case to $\alpha=\infty$.) To first order the relevant parameter is the correction rate α. The bias remaining at time t_n depends upon the product $\alpha t_n = wn^{\dagger}$, which can be interpreted as the "equivalent" number of radar corrections.

The subsequent analysis treats only the case of constant weight w. In practice w does vary, and even depends upon x. From the nature of the results, it appears likely that most of the important effects upon the guidance can be studied, at least qualitatively, through analysis of this simple case.

A.3 INITIAL VELOCITY ERROR $(\hat{b}_v(t))$

We first consider the effect of velocity filtering on the bias $\hat{c}(t)$ in the estimate of (vertical) velocity. The analysis is almost identical with that just given for position. Let c_0 be the initial velocity bias, let v be the constant weight used in filtering the radar velocity measurements, Δ_v

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†In general, upon $\sum_1^n w_i$ or $\int_0^t w(s)ds$.

the updating interval, $\beta \equiv v/\Delta_v$ the correction rate, and t_1 the time when velocity radar measurements are first processed.[†] Then, as in (18), the continuous approximation to the difference equation is

$$\hat{c}'(t) = \begin{cases} 0 , & 0 \leq t \leq t_1 \\ -\beta\hat{c}(t) , & t > t_1 \end{cases} \tag{22}$$

whose solution is

$$\hat{c}(t) = \begin{cases} c_0 & 0 \leq t \leq t_1 \\ c_0 e^{-\beta(t-t_1)} & t > t_1 \end{cases} \tag{23}$$

Equation (23) is applicable to all three velocity components (with appropriate β's).

The velocity bias also affects the extrapolated estimate of altitude, $\hat{x}^*$. Without any filtering, an initial velocity error c_0 results at time t in a position error $\hat{b}_v(t) = c_0(t)$. With altitude filtering, and with terrain bias $\tilde{b}$ assumed to be zero, we get from equation (11)

$$\frac{\hat{b}(t_{n+1})-\hat{b}(t_n)}{\Delta} = -\alpha\hat{b}(t_n) + (1-\overline{w})\hat{c}(t_n) \tag{24}$$

Letting $\Delta \to 0$ gives the differential equation

$$\hat{b}_v'(t) = -\alpha\hat{b}_v(t) + (1-\overline{w})\hat{c}(t) \tag{25}$$

[†]It is also assumed, as in the analysis of altitude bias, that accelerometer measurements are without error.

Assuming $\hat{b}(0) = 0$, the solution of (25) yields the velocity contribution to the altitude bias, viz.,

$$\hat{b}_v(t) = (1-\bar{w})e^{-\alpha t}\int_0^t e^{\alpha s}\,\hat{c}(s)ds \tag{26}$$

Substituting $\hat{c}(t)$ from equation (23) leads to explicit formulas for $\hat{b}_v(t)$. We consider first some special cases:

(i) $\bar{w} = 0 = w = \alpha$

$$\hat{b}_v(t) = \int_0^t \hat{c}(s)ds$$

$$= \begin{cases} c_o t\,, & 0 \leq t \leq t_1 \\ c_o t_1 + \frac{c_0}{\beta}\left(1-e^{-\beta(t-t_1)}\right)\,, & t > t_1 \end{cases} \tag{27}$$

(ii) $v = 0 = \beta$

$$\hat{c}(t) = c_0$$

$$\hat{b}_v(t) = (1-\bar{w})\,\frac{c_0}{\alpha}\,(1-e^{-\alpha t}) \tag{28}$$

(iii) $t_1 = 0$

$$\hat{c}(t) = c_0 e^{-\beta t}$$

$$\hat{b}_v(t) = (1-\bar{w})\,\frac{c_0}{\alpha-\beta}\,(e^{-\beta t} - e^{-\alpha t}),\ \alpha \neq \beta \tag{29a}$$

When $\alpha=\beta$, the last expression becomes indeterminate. Passing to the limit gives

$$\text{(iii')} \quad \hat{b}_V(t) = (1-\overline{w})c_0 e^{-\alpha t} \lim_{\alpha-\beta\to 0} \frac{e^{(\alpha-\beta)t}-1}{\alpha-\beta}$$

$$= (1-\overline{w})c_0 t e^{-\alpha t} \quad , \quad \alpha = \beta \qquad (29b)$$

(iv) General case. $\hat{c}(t)$ is given by (23) and

$$\frac{\hat{b}_V(t)}{(1-\overline{w})c_0} = \begin{cases} \frac{1}{\alpha}(1-e^{-\alpha t}), & 0 \le t \le t_1 \\ \frac{1}{\alpha}\left[\frac{\alpha e^{-\beta(t-t_1)} - \beta e^{-\alpha(t-t_1)}}{(\alpha-\beta)} - e^{-\alpha t}\right], & t > t_1,\ \alpha \ne \beta \\ (t-t_1)e^{-\alpha(t-t_1)} + \frac{1}{\alpha}(e^{-\alpha(t-t_1)} - e^{-\alpha t}), & t > t_1,\ \alpha = \beta \end{cases} \qquad (30)$$

A.4 <u>TERRAIN BIAS</u> ($\hat{b}_T(t)$)

The differential equation for the altitude bias due to terrain is obtained from equation (11) by setting $\hat{c}(t)=0$ and letting $\Delta\to 0$. (For simplicity the contribution due to computation delay δ, namely $\overline{w}\tilde{b}'(t)$, is omitted.) One gets

$$\hat{b}'_T(t) = -\alpha\tilde{b}_T(t) + \alpha\tilde{b}(t) \qquad (31)$$

where $\tilde{b}(t)$ describes the terrain variation. Including for the moment the contribution due to initial IMU bias b_0, the solution of (31) gives the altitude bias contributed by the terrain:

$$\hat{b}_T(t) = e^{-\alpha t}\left[b_0 + \int_0^t \alpha e^{\alpha s}\,\tilde{b}(s)ds\right] \qquad (32)$$

In the special case when $\tilde{b}(t) = B$, i.e., flat terrain with constant radar bias, and $b_0 = 0$, then

$$\hat{b}_T(t) = B(1-e^{-\alpha t}) \tag{33}$$

so that the full radar bias is eventually incorporated into the guidance system.

Equation (32) can be successively integrated by parts, assuming $\tilde{b}(t)$ to be sufficiently differentiable, to give*

$$\hat{b}_T(t) = \tilde{b}(t) + D\tilde{b}(t) - e^{-\alpha t}(\tilde{b}_0 - b_0 + D\tilde{b}_0) \tag{34}$$

where (for later use) we let the operator D represent derivatives up to the fourth order. Specifically,

$$D\tilde{b}(t) = \frac{-\tilde{b}'(t)}{\alpha} + \frac{\tilde{b}''(t)}{\alpha^2} - \frac{\tilde{b}'''(t)}{\alpha^3} + \frac{\tilde{b}^{(iv)}(t)}{\alpha^4} - R \tag{35}$$

$$\tilde{b}_0 = \tilde{b}(0) \quad , \quad D\,\tilde{b}_0 = D\tilde{b}(t)\Big|_{t=0} \tag{36}$$

$$R = \frac{e^{-\alpha t}}{\alpha^4} \int_0^t e^{\alpha s}\, \tilde{b}^{(v)}(s)\,ds \tag{37}$$

The three terms in (34) represent the three components of the filtered terrain bias (discussed in section 5.2 of the report), viz. (i) terrain alone, (ii) steady state filter bias, and (iii) filter transient.

We suppose now that the terrain has constant slope θ. If $z_D(=z(t_f))$ is the down range aim point, and if the terrain

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*If the effect of δ is included in (31), then $D\tilde{b}(t)$ and $D\tilde{b}_0$ are multiplied by $1-\bar{w}$.

bias is measured relative to the altitude at high gate, then

$$\tilde{b}(t) = \tan\theta\,[z_D - z(t)] \tag{38}$$

After the engine throttles down, the down-range component of the trajectory is the quartic polynomial

$$z(t) = z_D - \dot{z}_D(t_f-t) + \frac{1}{2}\ddot{z}_D(t_f-t)^2 + \frac{1}{6}c_{1z}(t_f-t)^3 + \frac{1}{12}c_{2z}(t_f-t)^4 \tag{39}$$

Note that z(t) is unperturbed by variations in the terrain. Equations (35) and (38) then imply that

$$D\tilde{b}(t) = -\tan\theta\ Dz(t) \tag{40}$$

From (35), (37) and (39) we get also

$$R = 0 \tag{41}$$

and

$$Dz(t) = -\frac{1}{\alpha}\sum_{i=0}^{3}\frac{a_i(t_f-t)^i}{i!} \tag{42}$$

where

$$
\begin{aligned}
a_3 &= -2c_{2z} \\
a_2 &= -c_{1z} + \frac{a_3}{\alpha} \\
a_1 &= -\ddot{z}_D + \frac{a_2}{\alpha} \\
a_0 &= \dot{z}_D + \frac{a_1}{\alpha}
\end{aligned}
\tag{43}
$$

Substituting (38) and (40) into (34) gives finally

$$\hat{b}_T(t) = b_0 e^{-\alpha t} + \tan\theta\,[z_D - z(t) - Dz(t) - e^{-\alpha t}(z_D - z_o - Dz_0)] \tag{44}$$

The initial and final conditions are to a certain extent ambiguous and several different cases are of interest. First, the bias can be referenced to the site rather than to high gate. In (34) we need only make the substitution

$$\tilde{b}(t) \rightarrow \tilde{b}(t) + \varepsilon_2 B \tag{45}$$

$$\tilde{b}_0 \rightarrow \tilde{b}_0 + \varepsilon_2 B \tag{46}$$

where

B = bias in terrain at high gate relative to the site†

$$\varepsilon_2 = \begin{cases} 0 \text{ if reference altitude is high gate} \\ 1 \text{ if reference altitude is the site} \end{cases} \tag{47}$$

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†Note that B is equivalent to a constant radar bias (cf. (33)). If the same slope θ continues from high gate to the site, then $B = -z_D \tan\theta$.

Secondly, if one assumes an initial IMU error of zero, this could be referenced either to the final terrain or to the initial terrain, i.e., either $b_0 = 0$ or $b_0 = \tilde{b}_0$. (An interpretation of the latter is that IMU and radar agree until departure from flat terrain occurs.) To accommodate this dual situation we can make the substitution

$$\tilde{b}_0 - b_0 \rightarrow \varepsilon_1 \tilde{b}_0 \tag{48}$$

where

$$\varepsilon_1 = \begin{cases} 0 & \text{if } b_0 = \tilde{b}_0 \\ 1 & \text{if } b_0 = 0 \end{cases} \tag{49}$$

More generally, let Δ_0 be the initial discrepancy between radar and IMU, i.e.,

$$\Delta_0 = \tilde{b}_0 - b_0 \tag{50a}$$

In terms of ε_1 and ε_2, (46) and (48) gives

$$\begin{aligned} \Delta_0 &= \varepsilon_1(\tilde{b}_0 + \varepsilon_2 B) \\ &= \varepsilon_1[\varepsilon_2 B + \tan\theta(z_D - z_o)] \end{aligned} \tag{50b}$$

Substituting (45) and (50) into (34) gives

$$\hat{b}_T(t) = \tilde{b}(t) + \varepsilon_2 B + D\tilde{b}(t) - e^{-\alpha t}(\Delta_0 + D\tilde{b}_0) \tag{51a}$$

In terms of z(t), this becomes

$$\hat{b}_T(t) = \varepsilon_2 B + \tan\theta[z_D - z(t) - Dz(t)] - e^{-\alpha t}[\varepsilon_1 \varepsilon_2 B + \tan\theta\{\varepsilon_1(z_D - z_o) - Dz_0\}] \quad (51b)$$

When $\varepsilon_1 = \varepsilon_2 = 0$, which is the situation of most interest, (51b) reduces to

$$\hat{b}_T(t) = \tan\theta[z_D - z(t) - Dz(t) - e^{-\alpha t}(-Dz_o)] \quad (51c)$$

We now suppose, more generally, that the terrain can be approximated by n sections each with constant slope, say $\theta_1, \ldots, \theta_n$. Let the slope θ_i occur for $z_{i-1} < z \leq z_i$, and let the corresponding time interval for the nominal z trajectory be $t_{i-1} < t \leq t_i$, where we set $t_0 = 0$, $t_n = t_f$. Let $\tilde{b}_i = \tilde{b}(t_i)$ be the terrain bias when $t = t_i$. Then, for $t_{i-1} < t \leq t_i$ (i=1,...,n)

$$\tilde{b}(t) = \tilde{b}_i + \tan\theta_i(z_i - z(t)) \quad (52)$$

When $\tilde{b}(t)$ is continuous (no jumps in the terrain), (52) implies that

$$\tilde{b}_{i-1} = \tilde{b}_i + \tan\theta_i(z_i - z_{i-1}) \quad (53)$$

I.e., the $\tilde{b}_i$ are computed backwards starting from high gate z_n $(=z_D)$, with $\tilde{b}_n = \varepsilon_2 B$ the bias at high gate.

A recursion relation for $\hat{b}_T(t)$, which generalizes (34), can be derived by subdividing the interval of integration in (32) at t_{i-1}. For $t_{i-1} < t \leq t_i$, we get

$$\hat{b}_T(t) = \tilde{b}(t) + D\tilde{b}(t) - e^{-\alpha(t-t_{i-1})}(\tilde{b}_{i-1}-\hat{b}_T(t_{i-1})+D\tilde{b}_{i-1}) \quad (54)$$

where $\hat{b}_T(0) = b_0$. Of some incidental interest is a symmetric form of (54):

$$e^{\alpha t}[\tilde{b}(t)-\hat{b}_T(t)+D\tilde{b}(t)] = e^{\alpha t_{i-1}}[\tilde{b}_{i-1}-\hat{b}_T(t_{i-1})+D\tilde{b}_{i-1}] \quad (55)$$

Substituting (52) and (53) into (54), gives the following recursive generalization of (44) and (51b)

$$\hat{b}_T(t) = \hat{b}_T(t_{i-1})e^{-\alpha(t-t_{i-1})}+\tilde{b}_i(1-e^{-\alpha(t-t_{i-1})}) + \tan\theta_i[z_i-z(t)-Dz(t) - e^{-\alpha(t-t_{i-1})}(z_i-z_{i-1}-Dz_{i-1})] \quad (56)$$

for $t_{i-1} < t \le t_i$. By induction it follows that

$$\hat{b}_T(t) = \tilde{b}_i+\tan\theta_i[z_i-z(t)-Dz(t)]-e^{-\alpha(t-t_{i-1})}[(\tan\theta_i-\tan\theta_{i-1})(-Dz_{i-1})] - e^{-(t-t_{i-2})}[(\tan\theta_{i-1}-\tan\theta_{i-2})(-Dz_{i-2})]-\ldots-e^{-\alpha t}[\tan\theta_1(-Dz_0)+\Delta_0] \quad (57)$$

Of particular interest is the special case where n=2 and θ_2=0 (i.e., slope θ for $0 < t \le t_1$ and flat terrain for $t_1 < t \le t_f$). Equation (57) then reduces to

$$\hat{b}_T(t) = \begin{cases} \varepsilon_2 B + \tan\theta\,[z_1 - z(t) - Dz(t)] - e^{-\alpha t}[\tan\theta(-Dz_0) + \Delta_0], & 0 < t \leq t_1, \\ \varepsilon_2 B + e^{-\alpha(t-t_1)}\tan\theta(-Dz_1) - e^{-\alpha t}[\tan\theta(-Dz_0) + \Delta_0], & t_1 < t \leq t_f \end{cases} \tag{58}$$

where

$$\Delta_0 = \varepsilon_1[\varepsilon_2 B + \tan\theta(z_1 - z_0)]$$

Although most terrain profiles can be approximated satisfactorily by a sequence of piece-wise constant slopes, a direct solution may also be of interest. Let $h(z)$ represent the altitude profile so that the radar bias is $\tilde{b}(z) = h(z_D) - h(z)$. The instantaneous slope is then $m(z) \equiv h'(z) = -b'(z)$. Integrating equation (32) successively by parts gives (where $z=z(t)$ or $z(s)$)

$$\hat{b}_T(t) = \tilde{b}(z) + e^{-\alpha t}\left(\int_0^t e^{\alpha s} m(z)\dot{z}(s)ds - \Delta_0\right) \tag{59}$$

$$= \tilde{b}(z) + m(z)Dz(t) - [m(z_0)Dz_0 + \Delta_0]\; e^{-\alpha t} +$$

$$+ e^{-\alpha t}\int_0^t e^{\alpha s} m'(z)\dot{z}(s)Dz(s)ds \tag{60}$$

APPENDIX B

ANALYSIS OF GUIDANCE PERTURBATIONS

B.1 PRELIMINARIES

The guidance differential equation (D.E.) for altitude is given by

$$\ddot{x}(t) = \ddot{x}_D - \frac{6}{t_f - t}(\dot{x}(t) + \dot{x}_D) - \frac{12}{(t_f - t)^2}(x(t) - x_D), \quad 0 \leq t \leq t_f \tag{1}$$

Assuming that t_f is constant, (1) is a Cauchy-type linear D.E. which can be converted into constant coefficients by the transformation

$$e^{-T} = 1 - \frac{t}{t_f}, \quad 0 \leq T < \infty \tag{2}$$

Letting ' represent d/dT, (1) becomes

$$x''(T) + 7x'(T) + 12x(T) = t_f^2 \ddot{x}_D - 6t_f \dot{x}_D + 12x_D \tag{3}$$

The eigenvalues are -4 and -3, so that e^{-4T} and e^{-3T} are independent solutions of the homogeneous portion of (3). The general solution is easily seen to be

$$x(T) = c_1 e^{-3T} + c_2 e^{-4T} + x_D - \dot{x}_D t_f e^{-T} + \frac{\ddot{x}_D}{2} t_f^2 e^{-2T} \tag{4}$$

In terms of $t_{go} = t_f - t$, we have, using (2),†

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†Although slightly ambiguous mathematically, it is more convenient to use t_{go} as the independent variable on the right side, and t on the left. If, say, $x^*(t_{go}) = x(t) = x(t_f - t_{go})$ and $' = d/dt_{go}$, then $\dot{x}(t) = -x^{*\prime}(t_{go})$, $\ddot{x}(t) = x^{*\prime\prime}(t_{go})$, etc.

$$x(t) = x_D - \dot{x}_D t_{go} + \frac{\ddot{x}_D}{2} t_{go}^{\ 2} + \frac{c_{1x}}{6} t_{go}^{\ 3} + \frac{c_{2x}}{12} t_{go}^{\ 4} \tag{5}$$

Differentiating (5) successively gives

$$\dot{x}(t) = \dot{x}_D - \ddot{x}_D t_{go} - \frac{c_{1x}}{2} t_{go}^{\ 2} - \frac{c_{2x}}{3} t_{go}^{\ 3} \tag{6}$$

$$\ddot{x}(t) = \ddot{x}_D + c_{1x} t_{go} + c_{2x} t_{go}^{\ 2} \tag{7}$$

From (5) - (7) one gets $x(t_f) = x_D$, $\dot{x}(t_f) = \dot{x}_D$, $\ddot{x}(t_f) = \ddot{x}_D$. If x_0 and $\dot{x}_0$ are the initial values for t=0, then (5) and (6) yield

$$c_{1x} = \frac{6}{t_f}\left[-\ddot{x}_D + \frac{(\dot{x}_0 + 3\dot{x}_D)}{t_f} + \frac{4(x_0 - x_D)}{t_f^{\ 2}}\right] \tag{8}$$

$$c_{2x} = \frac{12}{t_f^{\ 2}}\left[\frac{1}{2}\ddot{x}_D - \frac{(\dot{x}_0 + 2\dot{x}_D)}{t_f} - \frac{3(x_0 - x_D)}{t_f^{\ 2}}\right] \tag{9}$$

For purpose of analyzing disturbances to the D.E. (1) it is convenient to work with the state transition matrix ϕ. Since $t_{go}^{\ 3}$ and $t_{go}^{\ 4}$ are known to be independent solutions for $x(t)$, and $-3t_{go}^{\ 2}$ and $-4t_{go}^{\ 3}$ the corresponding solutions for $\dot{x}(t)$, ϕ can be obtained from the following equation (letting $s_{go} = t_f - s$)

$$\phi(t_{go},s_{go}) = \begin{pmatrix} t_{go}^{3} & t_{go}^{4} \\ -3t_{go}^{2} & -4t_{go}^{3} \end{pmatrix} \begin{pmatrix} s_{go}^{3} & s_{go}^{4} \\ -3s_{go}^{2} & -4s_{go}^{3} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 4\left(\frac{t_{go}}{s_{go}}\right)^3 - 3\left(\frac{t_{go}}{s_{go}}\right)^4 & s_{go}\left[\left(\frac{t_{go}}{s_{go}}\right)^3 - \left(\frac{t_{go}}{s_{go}}\right)^4\right] \\ \frac{12}{s_{go}}\left[\left(\frac{t_{go}}{s_{go}}\right)^3 - \left(\frac{t_{go}}{s_{go}}\right)^2\right] & 4\left(\frac{t_{go}}{s_{go}}\right)^3 - 3\left(\frac{t_{go}}{s_{go}}\right)^2 \end{pmatrix} \tag{10}$$

If the expressions in the first row are denoted by $\phi_1(t_{go}, s_{go})$ and $\phi_2(t_{go}, s_{go})$ respectively, and if $\dot{\phi}_1(t_{go}, s_{go}) = \frac{d}{dt}\phi_1(t_{go}, s_{go})$, and similarly for ϕ_2, then

$$\phi(t_{go}, s_{go}) = \begin{pmatrix} \phi_1(t_{go}, s_{go}) & \phi_2(t_{go}, s_{go}) \\ \dot{\phi}_1(t_{go}, s_{go}) & \dot{\phi}_2(t_{go}, s_{go}) \end{pmatrix} \tag{11}$$

For the non-homogeneous D.E.

$$\ddot{x}(t) + \frac{6\dot{x}(t)}{t_{go}} + \frac{12x(t)}{t_{go}^{2}} = \hat{f}(t) \quad (=f(t_{go})) \tag{12}$$

where $f(t_{go}) = \hat{f}(t_f - t_{go})$, the general solution can be obtained from the usual variation of parameters formula. Since we are interested primarily in $\ddot{x}(t)$, it is convenient to augment the state vector and transition matrix. Observing that

$$\frac{d}{dt}\int_{t_{go}}^{t_f} \dot{\phi}_2(t_{go},s_{go})f(s_{go})ds_{go}) = f(t_{go}) + \int_{t_{go}}^{t_f} \ddot{\phi}_2(t_{go},s_{go})f(s_{go})ds_{go} \quad ,$$

the augmented form of the solution can be written as follows.

$$\begin{pmatrix} x(t) \\ x(t) \\ \ddot{x}(t) \end{pmatrix} = \begin{pmatrix} \phi_1(t_{go},t_f) & \phi_2(t_{go},t_f) \\ \ddot{\phi}_1(t_{go},t_f) & \dot{\phi}_2(t_{go},t_f) \\ \ddot{\phi}_1(t_{go},t_f) & \ddot{\phi}_2(t_{go},t_f) \end{pmatrix} \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix} + \int_{t_{go}}^{t_f} f(s_{go}) \begin{pmatrix} \phi_2(t_{go},s_{go}) \\ \dot{\phi}_2(t_{go},s_{go}) \\ \ddot{\phi}_2(t_{go},s_{go}) \end{pmatrix} ds_{go}$$

$$+ \begin{pmatrix} 0 \\ 0 \\ f(t_{go}) \end{pmatrix} \qquad (13)$$

When $f(t_{go})$ corresponds to the original D.E. (1), i.e.,

$$f(t_{go}) = \ddot{x}_D - \frac{6x_D}{t_{go}} + \frac{12x_D}{t_{go}^2} \qquad (14)$$

the solution is equivalent to (5) - (9).

B.2 PERTURBATIONS DUE TO BIAS

In practice the guidance D.E. uses estimates, $\hat{x}$ and $\hat{\dot{x}}$, rather than true values of position and velocity. Equation (1) is then replaced by

$$\ddot{x}(t) = \ddot{x}_D - \frac{6}{t_{go}}(\hat{\dot{x}}(t)+\dot{x}_D) = \frac{12}{t_{go}^2}(\hat{x}(t)-x_D) \qquad (15)$$

Using the definitions of position and velocity bias in equations (A4) and (A6),† viz.

$$\hat{b}(t) = \hat{x}(t) - x(t) \tag{16}$$

$$\hat{c}(t) = \hat{\dot{x}}(t) - \dot{x}(t) \tag{17}$$

and expressing these in terms of t_{go}, i.e.,

$$b(t_{go}) = \hat{b}(t_f - t_{go}); \quad b(t_f) = \hat{b}(0) = b_0 \tag{18}$$

$$c(t_{go}) = \hat{c}(t_f - t_{go}); \quad c(t_f) = \hat{c}(0) = c_0 \tag{19}$$

equation (15) can be written as

$$\ddot{x}(t) - \ddot{x}_D + \frac{6}{t_{go}}(\dot{x}(t) + \dot{x}_D) + \frac{12}{t_{go}^2}(x(t) - x_D) = -\frac{12b(t_{go})}{t_{go}^2} - \frac{6c(t_{go})}{t_{go}} \tag{20}$$

The solution of (20) can be expressed as a perturbation of the "nominal" solution say $x_{nom}(t)$, obtained when the biases on the right had side are zero and the initial conditions are specified as $x_{nom,0}$ and $\dot{x}_{nom,0}$. (See Figure A1.) Define

$$\begin{aligned} \delta x(t) &= x(t) - x_{nom}(t) \\ \delta \dot{x}(t) &= \dot{x}(t) - \dot{x}_{nom}(t) \\ \delta \ddot{x}(t) &= \ddot{x}(t) - \ddot{x}_{nom}(t) \end{aligned} \tag{21}$$

and, in particular,

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†I.e., equations (4) and (6) in Appendix A.

$$\begin{aligned}\delta x_0 &= x_0 - x_{nom,0}\\ \delta \dot{x}_0 &= \dot{x}_0 - \dot{x}_{nom,0}\end{aligned} \tag{22}$$

When the initial estimated position corresponds to the nominal, i.e., $\hat{x}_0 = x_{nom,0}$, then (16) and (22) imply that $\delta x_0 = -b_0$. Similarly if $\hat{\dot{x}}_0 = \dot{x}_{nom,0}$, then $\delta \dot{x}_0 = -c_0$.

Using (21) and (22), the general solution (13) can now be expressed as a perturbation

$$\begin{aligned}\begin{pmatrix}\delta x(t)\\ \delta \dot{x}(t)\\ \delta \ddot{x}(t)\end{pmatrix} = &\begin{pmatrix} 4\left(\frac{t_{go}}{t_f}\right)^3 - 3\left(\frac{t_{go}}{t_f}\right)^4 & t_f\left[\left(\frac{t_{go}}{t_f}\right)^3 - \left(\frac{t_{go}}{t_f}\right)^4\right] \\ \frac{12}{t_f}\left[\left(\frac{t_{go}}{t_f}\right)^3 - \left(\frac{t_{go}}{t_f}\right)^2\right] & 4\left(\frac{t_{go}}{t_f}\right)^3 - 3\left(\frac{t_{go}}{t_f}\right)^2 \\ \frac{12}{t_f^2}\left[2\left(\frac{t_{go}}{t_f}\right) - 3\left(\frac{t_{go}}{t_f}\right)^2\right] & \frac{6}{t_f}\left[\left(\frac{t_{go}}{t_f}\right) - 2\left(\frac{t_{go}}{t_f}\right)^2\right]\end{pmatrix}\begin{pmatrix}\delta x_0\\ \delta \dot{x}_0\end{pmatrix} \; + \\ &+ \int_{t_{go}}^{t_f}\left[-12\frac{b(s_{go})}{s_{go}^2} - 6\frac{c(s_{go})}{s_{go}}\right]\begin{pmatrix} s_{go}\left[\left(\frac{t_{go}}{s_{go}}\right)^3 - \left(\frac{t_{go}}{s_{go}}\right)^4\right] \\ 4\left(\frac{t_{go}}{s_{go}}\right)^3 - 3\left(\frac{t_{go}}{s_{go}}\right)^2 \\ \frac{6}{s_{go}}\left[\left(\frac{t_{go}}{s_{go}}\right) - 2\left(\frac{t_{go}}{s_{go}}\right)^2\right]\end{pmatrix} ds_{go} \\ &+ \begin{pmatrix} 0 \\ 0 \\ -\frac{12b(t_{go})}{t_{go}^2} - \frac{6c(t_{go})}{t_{go}}\end{pmatrix}\end{aligned} \tag{23}$$

The first term on the right represents the effect of a change in initial conditions, while the remaining terms represent the perturbation due to the biases.

For ease of reference, we rewrite the expression for $\delta\ddot{x}$:

$$\delta\ddot{x}(t) = \frac{-12}{t_f^2}\left[3\left(\frac{t_{go}}{t_f}\right)^2 - 2\left(\frac{t_{go}}{t_f}\right)\right]\delta x_0 + 72\int_{t_{go}}^{t_f}\left(2\frac{t_{go}^2}{s_{go}^5} - \frac{t_{go}}{s_{go}^4}\right) b(s_{go})ds_{go}$$

$$- \frac{6}{t_f}\left[2\left(\frac{t_{go}}{t_f}\right)^2 - \left(\frac{t_{go}}{t_f}\right)\right]\delta\dot{x}_0 + 36\int_{t_{go}}^{t_f}\left(2\frac{t_{go}}{s_{go}^4} - \frac{t_{go}}{s_{go}^3}\right) c(s_{go})ds_{go}$$

$$- \frac{12b(t_{go})}{t_{go}^2} - \frac{6c(t_{go})}{t_{go}} \qquad (24)$$

Equation (24) can be put into more useful form by integrating by parts. Assuming $b(t_{go})$ and $c(t_{go})$ to be differentiable, one gets

$$\delta\ddot{x}(t) = -\frac{12}{t_f^2}\left[3\left(\frac{t_{go}}{t_f}\right)^2 - 2\left(\frac{t_{go}}{t_f}\right)\right](\delta x_0+b_0) + 12\int_{t_{go}}^{t_f}\left[\frac{3t_{go}^2}{s_{go}^4} - \frac{2t_{go}}{s_{go}^3}\right]$$

$$b'(s_{go})ds_{go} - \frac{6}{t_f}\left\{\left[2\left(\frac{t_{go}}{t_f}\right)^2 - \left(\frac{t_{go}}{t_f}\right)\right]\delta\dot{x}_0\right. \qquad (25)$$

$$\left. + \left[4\left(\frac{t_{go}}{t_f}\right)^2 - 3\left(\frac{t_{go}}{t_f}\right)\right]\cdot c_0\right\} + 6\int_{t_{go}}^{t_f}\left[\frac{4t_{go}^2}{s_{go}^3} - \frac{3t_{go}}{s_{go}^2}\right] c'(s_{go})ds_{go}$$

Similarly, one gets for δx and $\delta\dot{x}$

$$\delta x(t) = \left[4\left(\frac{t_{go}}{t_f}\right)^3 - 3\left(\frac{t_{go}}{t_f}\right)^4\right](\delta x_0 + b_0) - b(t_{go}) + \int_{t_{go}}^{t_f}\left[3\left(\frac{t_{go}}{s_{go}}\right)^4 - 4\left(\frac{t_{go}}{s_{go}}\right)^3\right] b'(s_{go})ds_{go} + \left[\left(\frac{t_{go}}{t_f}\right)^3 - \left(\frac{t_{go}}{t_f}\right)^4\right]\delta\dot{x}_0 t_f + \left[3\left(\frac{t_{go}}{t_f}\right)^3 - 2\left(\frac{t_{go}}{t_f}\right)^4\right] c_o t_f - t_{go}c(t_{go}) + \int_{t_{go}}^{t_f}\left(2\frac{t_{go}^4}{s_{go}^3} - 3\frac{t_{go}^3}{s_{go}^2}\right) c'(s_{go})\, ds_{go} \tag{26}$$

$$\delta\dot{x}(t) = -\frac{12}{t_f}\left[\left(\frac{t_{go}}{t_f}\right)^2 - \left(\frac{t_{go}}{t_f}\right)^3\right](\delta x_0 + b_0) + 12\int_{t_{go}}^{t_f}\left(\frac{t_{go}^2}{s_{go}^3} - \frac{t_{go}^3}{s_{go}^4}\right) b'(s_{go})ds_{go} + \left[4\left(\frac{t_{go}}{t_f}\right)^3 - 3\left(\frac{t_{go}}{t_f}\right)^2\right]\delta\dot{x}_0 - \left[9\left(\frac{t_{go}}{t_f}\right)^2 - 8\left(\frac{t_{go}}{t_f}\right)^3\right]c_0 + c(t_{go}) + \int_{t_{go}}^{t_f}\left(\frac{9t_{go}^2}{s_{go}^2} - \frac{8t_{go}^3}{s_{go}^3}\right) c'(s_{go})ds_{go} \tag{27}$$

By setting time-to go equal to the preassigned value used to mark termination of the braking phase, these equations yield the perturbation in high gate conditions.

Of some mathematical interest are the final values, $\delta x(t_f)$ and $\delta \dot{x}(t_f)$, corresponding to $t_{go}=0$. It is easy to show that if $n>0$, $m<n+1$, and $g(t_{go})$ is integrable and also bounded in a neighborhood of $t_{go}=0$, then

$$\lim_{t_{go}\to 0} \int_{t_{go}}^{t_f} \frac{t_{go}^{\,n}}{s_{go}^{\,m}}\, g(s_{go})\,ds_{go} = 0 \tag{28}$$

(26) then implies that

$$\delta x(t_f) = -b(0) \equiv -\hat{b}(t_f) \tag{29}$$

Similarly, (27), after integrating the first integral by parts, leads to

$$\begin{aligned} \delta \dot{x}(t_f) &= 2b'(0) + c(0) \\ &= -2\hat{b}'(t_f) + \hat{c}(t_f) \end{aligned} \tag{30}$$

while (25) gives

$$\begin{aligned} \delta \ddot{x}(t_f) &= -6(b''(0) + c'(0)) \\ &= -6(\hat{b}''(t_f) - \hat{c}'(t_f)) \end{aligned} \tag{31}$$

An alternative derivation of (29) - (31) is to expand $b(t_{go})$ and $c(t_{go})$ in (20) into a Taylor's series about $t_{go}=0$, and then transpose to the left hand side the terms in $t_{go}^{\,-2}$, $t_{go}^{\,-1}$, and $t_{go}^{\,0}$.

B.3 INITIAL ERROR IN ALTITUDE

Suppose that there is no velocity bias, i.e., $c(t_{go}) \equiv 0$, and that the altitude bias $b(t_{go})$ results from filtering (with constant weights) of an initial altitude error b_0. From equation (A20)

$$b'(t_{go}) = b_o \alpha e^{-\alpha t_f} e^{\alpha t_{go}} \tag{32}$$

Substituting into equation (25) gives

$$\delta\ddot{x}(t) = \frac{-12}{t_f^2}\left[3\left(\frac{t_{go}}{t_f}\right)^2 - 3\left(\frac{t_{go}}{t_f}\right)\right](\delta x_0 + b_0 + 12 b_0 \alpha e^{-\alpha t_f}$$

$$\left[3t_{go}^2 \int_{t_{go}}^{t_f} \frac{e^{\alpha s_{go}}}{s_{go}^4}\, ds_{go} - 2t_{go}\int_{t_{go}}^{t_f} \frac{e^{\alpha s_{go}}}{s_{go}^3}\, ds_{go}\right]$$

$$= \frac{12}{t_f^2}\left[3\left(\frac{t_{go}}{t_f}\right)^2 - 2\,\frac{t_{go}}{t_f}\right] \cdot (\delta x_0 + b_0) +$$

$$12 b_0 \alpha^2 e^{-\alpha t_f}\left[3(\alpha t_{go})^2 \int_{\alpha t_{go}}^{\alpha t_f} \frac{e^x}{x^4}\, dx - 2(\alpha t_{go})\int_{\alpha t_{go}}^{\alpha t_f} \frac{e^x}{x^3}\, dx\right] \tag{33}$$

The following recurrence relation is readily obtained upon integrating by parts:

$$\int_{\alpha t_{go}}^{\alpha t_f} \frac{e^x}{x^n}\, dx = \frac{1}{n-1}\left[\frac{e^{\alpha t_{go}}}{(\alpha t_{go})^{n-1}} - \frac{e^{\alpha t_f}}{(\alpha t_f)^{n-1}} + \int_{\alpha t_{go}}^{\alpha t_f} \frac{e^x}{x^{n-1}}\, dx\right] \tag{34}$$

By successive application of (34) the integral on the left can be evaluated, for n an integer, in terms of the exponential integral†

$$\mathrm{Ei}(\xi,\eta) = \int_{\xi}^{\eta} \frac{e^x}{x}\,dx \tag{35}$$

$$= \mathrm{Ei}(\eta) - \mathrm{Ei}(\xi)$$

In particular, equation (33) can be written

$$\delta\ddot{x}(t) = \frac{12}{t_f^2}\left[2\left(\frac{t_{go}}{t_f}\right)^2 - 3\left(\frac{t_{go}}{t_f}\right)^2\right](\delta x_0 + b_0) + \frac{12b_0}{t_f^2}F(\alpha t_{go},\alpha t_f) \tag{36}$$

where

$$F(\xi,\eta) = \eta^2 e^{-\eta}\left[3\xi^2\int_{\xi}^{\eta}\frac{e^x}{x^4}\,dx - 2\xi\int_{\xi}^{\eta}\frac{e^x}{x^3}\,dx\right] \tag{37}$$

$$= \xi - \frac{\xi^2}{\eta} + \eta^2 e^{-\eta}\left[\xi^2\int_{\xi}^{\eta}\frac{e^x}{x^3}\,dx - \xi\int_{\xi}^{\eta}\frac{e^x}{x^2}\,dx\right]$$

$$= \xi - \frac{\xi^2}{\eta} + \frac{\eta}{2}\left[\eta e^{-(\eta-\xi)}(\xi-1) + 2\xi - \xi^2\left(1+\frac{1}{\eta}\right) + \eta e^{-\eta}\xi(-2)\mathrm{Ei}(\xi,\eta)\right] \tag{38}$$

When $w = 0 = \alpha$

$$\delta\ddot{x}(t) = \frac{12}{t_f^2}\left(2\,\frac{t_{go}}{t_f} - 3\,\frac{t_{go}^2}{t_f^2}\right)(\delta x_0 + b_0) \tag{39}$$

so that $\delta\ddot{x}(t) \equiv 0$ when $\delta x_0 = -b_0$. When $w=1$ ($\alpha=\infty$), i.e., the entire bias b_0 is removed at $t=0$, then $b(t_{go}) = 0$ for $0 \lesssim t_{go} < t_f$ and equation (24) gives

† In the computations, values of $\mathrm{Ei}(\eta) = \mathrm{Ei}(-\infty,\eta)$, for η an integer, were taken from the tabulations in References 7 and 8. For non-integral η, the excess was obtained by numerical integration. Several checks showed accuracy to seven significant digits.

$$\delta\ddot{x}(t) = \frac{12}{t_f^2}\left(2\,\frac{t_{go}}{t_f} - 3\,\frac{t_{go}^2}{t_f^2}\right)\delta x_0$$

$$= \frac{12b_0}{t_f^2}\left(3\,\frac{t_{go}^2}{t_f^2} - 2\,\frac{t_{go}}{t_f}\right) \qquad (40)$$

(39) and (40) are compatible with (36), since one can show that

$$\lim_{\alpha\to\infty} F(\alpha t_{go},\alpha t_f) = \begin{cases} 3\left(\dfrac{t_{go}}{t_f}\right)^2 - 2\left(\dfrac{t_{go}}{t_f}\right), & \text{for } 0 \le t_{go} < t_f \\ 0, & \text{for } t_{go} = t_f \end{cases} \qquad (41)$$

$$\lim_{\alpha\to 0} F(\alpha t_{go},\alpha t_f) = 0 \qquad , \text{ for } 0 \le t_{go} \le t_f$$

The final value of perturbation acceleration, $\delta\ddot{x}(t_f)$, can be derived from (36) since (28) implies that $\lim_{\xi\to 0} \xi Ei(\xi,\eta) = 0$ so that from (38) $F(0,\eta) = 1/2\ \eta^2 e^{-\eta}$ and

$$\delta\ddot{x}(t_f) = -6b_0\ \alpha^2 e^{-\alpha t_f} \qquad (42)$$

This agrees with the general formula (31). Similarly the final values of position and velocity are, from (29) and (30),

$$\delta x(t_f) = -b(0)$$
$$= -b_0 e^{-\alpha t_f} \qquad (43)$$

$$\delta\dot{x}(t_f) = 2b'(0)$$
$$= 2b_0\alpha e^{-\alpha t_f} \qquad (44)$$

B.4 INITIAL ERROR IN VELOCITY

When the initial velocity is c_o, the velocity and altitude bias functions, $c(t_{go})$ and $b_v(t_{go})$, are given by equations (A23) and (A30). To determine the guidance response it is convenient to integrate by parts the general expression (25) for $\delta\ddot{x}(t)$. Assuming $\delta x_0 = 0$ and noting that $\hat{b}_v(0)(=b_v(t_f)) = 0$ and $\hat{b}_v'(0)(=b_v'(t_f)) = (1-\bar{w})c_o$, we get

$$\delta\ddot{x}(t) = \frac{6}{t_f}\left(\frac{t_{go}}{t_f} - \frac{2t_{go}^2}{t_f^2}\right)(\delta x_0+c_o) + \frac{12c_o\bar{w}}{t_f}\left(\frac{t_{go}}{t_f} - \frac{t_{go}^2}{t_f^2}\right)$$
$$+ 12\int_{t_{go}}^{t_f}\left(\frac{t_{go}}{s_{go}^2} - \frac{t_{go}^2}{s_{go}^3}\right)b_v''(s_{go})ds_{go}$$
$$+ 6\int_{t_{go}}^{t_f}\left(\frac{4t_{go}^2}{s_{go}^3} - \frac{3t_{go}}{s_{go}^2}\right)c'(s_{go})ds_{go} \qquad (45)$$

The overall result is seen to consist of the sum of an initial condition perturbation, a perturbation due to the induced altitude bias (depending upon both v and w), and a perturbation due to the velocity bias (depending only on v). $c(t_{go})$ is proportional to c_o, while $b_v(t_{go})$ is proportional to $(1-\bar{w})c_o$. Also the altitude filter is assumed to start at t=0 and the velocity filter at $t_1 (0 \le t_1 < t_f)$. Before treating the general case, it is useful to examine the special cases of w=0, v=0 and t_1=0, respectively.

4.1 Velocity Filtering Only (w=0)

Let the velocity bias function be $c(t_{go})$. When w=0, the altitude bias is

$$b_v(t_{go}) = \int_{t_{go}}^{t_f} c(s_{go})ds_{go}$$

so that

$$b_v''(t_{go}) = -c'(t_{go})$$

Hence in equation (45) the two integrals can be combined to give

$$\delta\ddot{x}(t) = \frac{6}{t_f}\left(\frac{t_{go}}{t_f} - 2\frac{t_{go}^2}{t_f^2}\right)(\delta\dot{x}_0+c_o) + \left[6\int_{t_{go}}^{t_f}\left(2\frac{t_{go}^2}{s_{go}^3} - \frac{t_{go}}{s_{go}^2}\right)c'(s_{go})ds_{go}\right] \tag{46}$$

With constant velocity filtering starting at t_1, we have (equation (A23))

$$c'(t_{go}) = c_o\beta e^{-\beta t_{f_1}}e^{\beta t_{go}} \cdot h(t_{f_1} - t_{go}) \tag{47}$$

where

$$t_{f_1} = t_f - t_1 \tag{48}$$

$$h(t_{f_1}-t_{go}) = \begin{cases} 0 \text{ for } t_f \geq t_{go} > t_{f_1} & (t<t_1) \\ 1 \text{ for } 0 \leq t_{go} \leq t_{f_1} & (t\geq t_1) \end{cases} \tag{49}$$

Equation (46) then becomes

$$\delta\ddot{x}(t) = \frac{6}{t_f}\left(\frac{t_{go}}{t_f} - 2\frac{t_{go}^2}{t_f^2}\right)(\delta\dot{x}_0+c_o) + \frac{6c_o}{t_{f_1}} f_{bc}(\beta t_{go},\beta t_{f_1})h(t_{f_1}-t_{go}) \tag{50}$$

where

$$f_{bc}(\xi,\eta) = e^{-\eta}\left(2\xi^2\int_\xi^\eta \frac{e^x}{x^3}\,dx - \xi\int_\xi^\eta \frac{e^x}{x^2}\,dx\right) \tag{51}$$

$$= \xi[\eta e^{-(\eta-\xi)}+1 - \xi - \frac{\xi}{\eta} + \eta e^{-\eta}(\xi-1)Ei(\xi,\eta)]$$

Under nominal conditions of $\delta\dot{x}_0 = -c_o$, the first term in (50) vanishes so that $\delta\ddot{x}(t) = 0$ for $0 \le t \le t_1$. When $v=1(\beta=\infty)$ for $t \ge t_1$ (and $v=0$ for $t<t_1$), $c(t)$ jumps at t_1 from c_o to 0 and one gets the limiting solution

$$\delta\ddot{x}(t) = \frac{6c_o}{t_{f_1}}\left(2\frac{t_{go}^2}{t_{f_1}^2} - \frac{t_{go}}{t_{f_1}}\right)\cdot h(t_{f_1} - t_{go}) \tag{52}$$

Note that the downrange component, $x(t)$, of the LM trajectory employs velocity filtering but not position filtering so that equation (50) is applicable to $\delta\ddot{z}(t)$ as well. However, in the present LM guidance program, perturbations in $z(t)$ will cause t_f to vary slightly and the D.E. is no longer strictly linear.

4.2 Altitude Filtering Only (v=0)

When v=0, $c'(t_{go}) = 0$, so that the last term in equation (45) is zero. From equation (A28) we have

$$b_v''(t_{go}) = -(1-\bar{w})\; c_o \alpha t_f e^{-\alpha t_f} e^{\alpha t_{go}} \tag{53}$$

Equation (45) then gives, with $\delta\dot{x}_0 = -c_o$,

$$\delta\ddot{x}(t) = \frac{12c_o}{t_f}\left[\bar{w}\left(\frac{t_{go}}{t_f} - \frac{t_{go}^2}{t_f^2}\right) + (1-\bar{w}) f_b(\alpha t_{go}, \alpha t_f)\right] \tag{54}$$

where

$$\begin{aligned} f_b(\xi,\eta) &= e^{-\eta}\left[\xi \int_\xi^\eta \frac{e^x}{x^2}\,dx - \xi^2 \int_\xi^\eta \frac{e^x}{x^3}\,dx\right] \\ &= -\frac{1}{2}\left[\eta e^{-(\eta-\xi)}(\xi-1)+2\xi - \frac{\xi^2}{\eta} - \xi^2 + \eta e^{-\eta}\xi(\xi-2)\mathrm{Ei}(\xi,\eta)\right] \end{aligned} \tag{55}$$

Comparing (55) with (38) shows that

$$f_b(\xi,\eta) = \frac{\xi}{\eta} - \frac{\xi^2}{\eta^2} - \frac{F(\xi,\eta)}{\eta} \tag{55a}$$

so that (54) can also be written

$$\delta\ddot{x}(t) = \frac{12c_o}{t_f}\left(\frac{t_{go}}{t_f} - \frac{t_{go}^2}{t_f^2}\right) - \frac{12(1-\bar{w})c_o/\alpha}{t_f^2} F(\alpha t_{go}, \alpha t_f) \tag{54a}$$

When w=1 ($\alpha=\infty$), the second term vanishes.

4.2 Both Velocity and Altitude Filtering ($t_1=0$)

When both altitude and velocity filtering begin at $t=0$, the bias functions in equations (A29) give

$$c'(t_{go}) = c_o \beta e^{-\beta t_f} e^{\beta t_{go}} \tag{56}$$

$$\frac{-b_v''(t_{go})}{(1-\bar{w})c_o} = \begin{cases} \frac{\alpha}{\alpha-\beta}\,(\alpha t_f e^{-\alpha t_f} e^{\alpha t_{go}}) - \frac{\beta}{\alpha-\beta}(\beta t_f e^{-\beta t_f} e^{\beta t_{go}}), & \text{for } \alpha\neq\beta \\ (2 - \alpha t_f + \alpha t_{go})\alpha t_f e^{-\alpha t_f} e^{\alpha t_{go}}, & \text{for } \alpha=\beta \end{cases} \tag{57}$$

$\delta\ddot{x}(t)$, from (45), is then given by

$$\delta\ddot{x}(t) = \frac{12c_o}{t_f}\left[\bar{w}\left(\frac{t_{go}}{t_f} - \frac{t_{go}^2}{t_f^2}\right) + (1-\bar{w})f_{b_o}(t_{go};\alpha,\beta)\right] + \frac{6c_o}{t_f}f_c(\beta t_{go},\beta t_f) \tag{58}$$

where

$$f_{b_o}(t_{go};\alpha,\beta) = \begin{cases} \frac{\alpha}{\alpha-\beta}\,f_b(\alpha t_{go},\alpha t_f) - \frac{\beta}{\alpha-\beta}\,f_b(\beta t_{go},\beta t_f) & \text{for } \alpha\neq\beta \\ (2-\alpha t_f)\cdot f_b(\alpha t_{go},\alpha t_f) + g_b(\alpha t_{go},\alpha t_f) & \text{for } \alpha=\beta \end{cases} \tag{59}$$

$f_b(\xi,\eta)$ is given by (55), and

$$g_b(\xi,\eta) = \eta e^{-\eta}\left(\xi \int_\xi^\eta \frac{e^x}{x}\,dx - \xi^2 \int_\xi^\eta \frac{e^x}{x^2}\,d_x\right)$$

$$= \xi[\eta e^{-(\eta-\xi)} - \xi + \eta e^{-\eta}(\xi-1)Ei(\xi,\eta)] \tag{60}$$

while

$$f_c(\xi,\eta) = \eta e^{-\eta}\left(4\xi^2 \int_\xi^\eta \frac{e^x}{x^3}\,dx \quad -3\xi \int_\xi^\eta \frac{e^x}{x^2}\,dx\right)$$

$$= \eta e^{-(\eta-\xi)}(2\xi-1) + 3\xi - 2\xi^2 - \frac{\xi^2}{\eta} + \eta e^{-\eta}\xi(2\xi-\beta)Ei(\xi,\eta) \tag{61}$$

When w=1 ($\alpha=\infty$), the term in brackets in (58) reduces to $[(t_{go}/t_f) - (t_{go}/t_f)^2]$. When v=1 ($\beta=\infty$), the limiting solution is

$$\delta\ddot{x}(t) = \frac{6c_o}{t_f}\left(2\frac{t_{go}^2}{t_f^2} - \frac{t_{go}}{t_f}\right) \tag{62}$$

This holds for all w.

4.4 Both Velocity and Altitude Filtering ($t_1>0$)

The bias functions are given in equations (A23) and (A30). $c'(t_{go})$ is the same as equation (47). It is convenient to divide the total altitude bias b_v into the portion due to w alone, say b_{v_o}, plus the remaining portion, b_{v_1}, which includes the effect of v starting at time t_1. That is, we can write

$$b_v(t_{go}) = b_{v_o}(t_{go}) + b_{v_1}(t_{go}) \cdot h(t_{f_1}-t_{go}) \tag{63a}$$

and hence also

$$b_v''(t_{go}) = b_{v_o}''(t_{go}) + b_{v_1}''(t_{go}) \cdot h(t_{f_1} - t_{go}) \tag{63b}$$

where, as in (53),

$$b_{v_o}''(t_{go}) = -(1-\bar{w})c_o \alpha t_f e^{-\alpha t_f} e^{\alpha t_{go}} \tag{64}$$

and

$$\frac{-b_{v_1}''(t_{go})}{(1-\bar{w})c_o} = \begin{cases} \frac{\beta}{\alpha-\beta}\left[\alpha t_{f_1} e^{-\alpha t_{f_1}} e^{\alpha t_{go}} - \beta t_{f_1} e^{-\beta t_{f_1}} e^{\beta t_{go}}\right], & \alpha \neq \beta \\ (1 - \alpha t_{f_1} + \alpha t_{go})\, \alpha t_{f_1} e^{-\alpha t_{f_1}} e^{\alpha t_{go}}, & \alpha = \beta \end{cases} \tag{65}$$

Equation (45) can then be written

$$\delta\ddot{x}(t) = \frac{12c_o}{t_f}\left[\bar{w}\left(\frac{t_{go}}{t_f} - \frac{t_{go}^2}{t_f^2}\right) + (1-\bar{w}) f_b(\alpha t_{go}, \alpha t_f)\right] + \left[\frac{12c_o}{t_{f_1}}(1-\bar{w}) \cdot f_{b_1}(t_{go};\alpha,\beta) + \frac{6c_o}{t_{f_1}} f_c(\beta t_{go}, \beta t_{f_1})\right] h(t_{f_1} - t_{go}) \tag{66}$$

where $f_b(\xi,\eta)$ and $f_c(\xi,\eta)$ are given by (55) and (61), while

$$f_{b_1}(t_{go};\alpha,\beta) = \begin{cases} \frac{\beta}{\alpha-\beta}\,[f_b(\alpha t_{go},\alpha t_{f_1}) - f_b(\beta t_{go},\beta t_{f_1})] & \text{for } \alpha\neq\beta \\ (1- t_{f_1})f_b(\alpha t_{go},\alpha t_{f_1}) + g_b(\alpha t_{go},\alpha t_{f_1}) & \text{for } \alpha=\beta \end{cases} \tag{67}$$

with $g_b(\xi,\eta)$ given by (60). When $t_1=0$, (66) reduces to (58).

One can obtain the limiting expression of (66) for $v=1$ $(\beta=\infty)$ from the relations

$$\lim_{\beta\to\infty} f_c(\beta t_{go},\beta t_{f_1}) = \left[4\left(\frac{t_{go}}{t_{f_1}}\right)^2 - 3\left(\frac{t_{go}}{t_{f_1}}\right)\right] \tag{68}$$

$$\lim_{\beta\to\infty} f_b(\beta t_{go},\beta t_{f_1}) = \left(\frac{t_{go}}{t_{f_1}}\right) - \left(\frac{t_{go}}{t_{f_1}}\right)^2 \tag{69}$$

The result is

$$\begin{aligned} \delta\ddot{x}(t) = {} & \frac{12c_o}{t_f}\left[\bar{w}\left(\frac{t_{go}}{t_f} - \frac{t_{go}^2}{t_f^2}\right) + (1-\bar{w})f_b(\alpha t_{go},\alpha t_f)\right] \\ & + h(t_{f_1}-t_{go})\left[\frac{6c_o}{t_{f_1}}\left(2\frac{t_{go}^2}{t_{f_1}^2} - \frac{t_{go}}{t_{f_1}}\right) - \frac{12c_o}{t_{f_1}}\left\{\bar{w}\left(\frac{t_{go}}{t_{f_1}} - \frac{t_{go}^2}{t_{f_1}^2}\right)\right.\right. \\ & \left.\left. + (1-\bar{w})f_b(\alpha t_{go},\alpha t_{f_1})\right\}\right] \end{aligned} \tag{70}$$

Equation (70) reduces, in special cases, to formulas previously derived--(52), (54) and (62). For $\overline{w}=1$ or $\alpha=\infty$, (70) reduces to

$$\delta\ddot{x}(t) = \frac{12c_o}{t_f}\left(\frac{t_{go}}{t_f} - \frac{t_{go}^2}{t_f^2}\right) + h(t_{f_1}-t_{go})\frac{6c_o}{t_{f_1}}\left(4\frac{t_{go}^2}{t_{f_1}^2} - 3\frac{t_{go}}{t_{f_1}}\right) \qquad (71)$$

B.5 BIAS DUE TO SLOPES

We consider first the case where the slope θ is constant over the entire interval, $0 \le t \le t_f$. In order to write the bias equations of Appendix A.4 in terms of t_{go}, we define range-to-go to high gate

$$z_g(t_{go}) = z_D - z(t_f-t_{go})$$

$$= \dot{z}_D t_{go} - \frac{1}{2}\ddot{z}_D t_{go}^2 - \frac{1}{6}c_{1z}t_{go}^3 - \frac{1}{12}c_{2z}t_{go}^4 \qquad (72)$$

so that

$$\begin{aligned} z_g'(t_{go}) &= z'(t_f-t_{go}) \\ z_g''(t_{go}) &= -z''(t_f-t_{go}) \end{aligned} \qquad (73)$$

and so on. The operator D (equations (A35), (A39) and (A42)) can be rewritten as

$$D_g z_g(t_{go}) \equiv -Dz(t_f-t_{go})$$

$$= \frac{z_g'(t_{go})}{\alpha} + \frac{z_g''(t_{go})}{\alpha^2} + \ldots + \frac{z_g^{(iv)}(t_{go})}{\alpha^4} \qquad (74)$$

$$= \frac{1}{\alpha}\sum_{i=0}^{3}\frac{a_i t_{go}^i}{i!} \qquad (75)$$

where the a_i are defined in (A43). Equations (A50b) and (A51b) then give

$$b_T'(t_{go}) = \tan\theta\left[z_g'(t_{go}) + D_g'z_g(t_{go}) - \alpha e^{\alpha t_{go}}e^{-\alpha t_f}\{\Delta_0\cot\theta + D_g z_g(t_f)\}\right] \tag{76}$$

where Δ_0, the initial altitude discrepancy, is

$$\begin{aligned}\Delta_0 &= \tilde{b}_0 - b_0 \\ &= \varepsilon_1(\varepsilon_2 B + z_g(t_f)\tan\theta)\end{aligned} \tag{77}$$

From (74) and (75), since $z_g^{(v)}(t_{go}) = 0$, we have

$$\begin{aligned}z_g'(t_{go}) + D_g'z_g(t_{go}) &= \alpha D_g z_g(t_{go}) \\ &= \sum_{i=0}^{3} \frac{a_i}{i!} t_{go}{}^i\end{aligned} \tag{78}$$

Defining also

$$\begin{aligned}D_o &= D_g z_g(t_f) \\ &= \frac{1}{\alpha}\sum_{i=0}^{3} \frac{a_i t_f{}^i}{i!}\end{aligned} \tag{79}$$

(76) can then be written as

$$b_T'(t_{go}) = \tan\theta \sum_{i=0}^{3} \frac{a_i t_{go}{}^i}{i!} - (\Delta_0\cot\theta + D_0)\alpha e^{-\alpha t_f}e^{\alpha t_{go}} \tag{80}$$

Substituting (80) into the general perturbation equation (25), with $\delta x_o = -b_o$, the final result can be written

$$\delta\ddot{x}(t) = 12\tan\theta\left[\sum_{i=0}^{3}\frac{a_i}{i!}\,\delta_i\ddot{x}(t_{go},t_f) - \frac{\Delta_0\cot\theta + D_o}{t_f^{\,2}}\,F(\alpha t_{go},\alpha t_f)\right] \tag{81}$$

where $F(\xi,\eta)$ is given by (38), and

$$\delta_i\ddot{x}(t_{go},t_f) = \int_{t_{go}}^{t_f}\left(\frac{3t_{go}^{\,2}}{s_{go}^{\,4}} - \frac{2t_{go}}{s_{go}^{\,3}}\right) s_{go}^{\,i}\,ds_{go} \tag{82}$$

Specifically,

$$\begin{aligned}
\delta_0\ddot{x}(t_{go},t_f) &= \frac{1}{t_f}\left(\frac{t_{go}}{t_f}\right)\left(1 - \frac{t_{go}}{t_f}\right)\\
\delta_1\ddot{x}(t_{go},t_f) &= \left(\frac{3}{2}\frac{t_{go}}{t_f} - \frac{1}{2}\right)\left(1 - \frac{t_{go}}{t_f}\right)\\
\delta_2\ddot{x}(t_{go},t_f) &= t_{go}\left(3 - 3\frac{t_{go}}{t_f} + 2\ln\frac{t_{go}}{t_f}\right)\\
\delta_3\ddot{x}(t_{go},t_f) &= -t_{go}t_f\left(1 - \frac{t_{go}}{t_f} + 3\frac{t_{go}}{t_f}\ln\frac{t_{go}}{t_f}\right)
\end{aligned} \tag{83}$$

The first term in (81) includes both the effect of terrain alone, and also the steady state filter effect; the last term is the transient. The terrain effect can be isolated by letting w=1 ($\alpha=\infty$). Since $D_g z_g(t_{go}) = 0 = D_0$, one gets simply

$$\ddot{\delta x}(t) = 12\tan\theta \int_{t_{go}}^{t_f} \left(\frac{3t_{go}^2}{s_{go}^4} - \frac{2t_{go}}{s_{go}^3}\right) z_g'(s_{go})ds_{go} - \frac{12\Delta_0}{t_f^2}\left(3\frac{t_{go}^2}{t_f^2} - 2\frac{t_{go}}{t_f}\right) \tag{84}$$

The terminal biases, obtained by substituting equation (A51b) into equations (29)-(31), are

$$\delta x(t_f) = -\tan\theta\left[\frac{a_o}{\alpha} - (\Delta_o\cot\theta + D_o)e^{-\alpha t_f}\right]$$

$$\delta\dot{x}(t_f) = 2\left[a_o - \alpha(\Delta_o\cot\theta + D_o)e^{-\alpha t_f}\right]\tan\theta$$

$$= -2\alpha\delta x(t_f)$$

$$\delta\ddot{x}(t_f) = -6\tan\theta\left[a_1 - \alpha^2(\Delta_o\cot\theta + D_o)e^{-\alpha t_f}\right]$$

$$= 6\left[\alpha^2\delta x(t_f) + \alpha\dot{z}_D\tan\theta\right]$$

It is of interest to examine the simple case where the down range trajectory is linear, i.e., with (72) replaced by

$$z_g(t_{go}) = \dot{z}_D \cdot t_{go}$$

(81) then becomes

$$\ddot{\delta x}(t) = \tan\theta\left[\frac{12\dot{z}_D}{t_f}\left(\frac{t_{go}}{t_f} - \frac{t_{go}^2}{t_f^2}\right) - \frac{12\dot{z}_D/\alpha}{t_f^2}F(\alpha t_{go},\alpha t_f)\right] - \frac{12\Delta_0}{t_f^2}F(\alpha t_{go},\alpha t_f) \tag{85}$$

$$= \frac{12\dot{z}_D\tan\theta}{t_f}f_b(\alpha t_{go},\alpha t_f) - \frac{12\Delta_0}{t_f^2}F(\alpha t_{go},\alpha t_f) \tag{86}$$

Comparing (85) with (54a) shows that when $\delta=0=\overline{w}$ and the horizontal component of motion has uniform velocity, then a constant terrain slope is equivalent to a velocity error $c_0 = \dot{z}_D \tan\theta$. Actually if equation (31) had included the term, $-\overline{w}\dot{z}_D$, representing the contribution of δ, the equivalence could be shown to hold for $\delta>0$ as well. It seems clear that the actual quartic trajectory with perturbation (81) corresponds to an (input) velocity error that varies with time according to a cubic polynomial.

The generalization to n slopes $\theta_1,\ldots,\theta_n$ is straightforward. The bias equation (A57) extends formula (80) to

$$b_T'(t_{go}) = \tan\theta_i \cdot \sum_{j=0}^{3} \frac{a_j t_{go}^{\,j}}{j!} - (\tan\theta_i - \tan\theta_{i-1}) D_{i-1} \alpha e^{-\alpha t_{f_{i-1}}} e^{\alpha t_{go}} - \qquad (87)$$

$$- \ldots - (\tan\theta_2 - \tan\theta_1) D_1 \alpha e^{-\alpha t_{f_1}} e^{\alpha t_{go}} - (D_0 \tan\theta_1 + \Delta_0) \alpha e^{-\alpha t_f} e^{\alpha t_{go}}$$

for $t_{f_i} \leq t_{go} < t_{f_{i-1}}$, where $t_{f_i} = t_f - t_i$ and

$$D_k = D_g z_g (t_{f_k}) \qquad (88)$$

$$= \frac{1}{\alpha} \sum_{j=0}^{3} \frac{a_j}{j!} t_{f_k}^{\,j}$$

The perturbation is obtained by substituting (87) into (25). The form of the final result can be simplified by defining

$$\delta_i * \ddot{x}(t_{go}, t_{f_i}) = \sum_{j=0}^{3} \frac{a_j}{j!} \delta_j x(t_{go}, t_{f_i}) - \frac{D_i}{t_{f_i}^{\,2}} F(\alpha t_{go}, \alpha t_{f_i}) \qquad (89)$$

Then

$$\delta\ddot{x}(t) = 12\sum_{i=0}^{n-1} h(t_{f_i}-t_{go})\delta_i{}^{*}\ddot{x}(t_{go},t_{f_i})(\tan\theta_{i+1}-\tan\theta_i) -$$

$$- 12\frac{\Delta_0}{t_f{}^2} F(\alpha t_{go},\alpha t_f) \tag{90}$$

where $t_{f_0}=t_f$ and $\theta_0=0$. Equation (90) states that the total perturbation at time t is obtained by superimposing the individual perturbations generated by all previous <u>incremental changes</u> in slope.

When n=2 and the slope θ occurs only for $0 \le t < t_1$ with the terrain being flat thereafter (i.e., $\theta_1=\theta$ and $\theta_2=0$), equation (90) reduces to

$$\delta\ddot{x}(t) = 12\tan\theta\left[\sum_{j=0}^{3}\frac{a_j}{j!}\delta_j x(t_{go},t_f) - \frac{D_0+\Delta_0\cot\theta}{t_f{}^2}F(\alpha t_{go},\alpha t_f) -\right.$$

$$\left. - h(t_{f_1}-t_{go})\left\{\sum_{j=0}^{3}\frac{a_j}{j!}\delta_j x(t_{go},t_{f_1}) - \frac{D_1}{t_{f_1}{}^2}F(\alpha t_{go},\alpha t_{f_1})\right\}\right] \tag{91}$$

Table 1. Terminal Biases Due to Initial Velocity Error[1]

$t_1 = 0$

V \ W	0	.05	.1	.2	1.0
0	-2400 -20 0	-722 18.1 .14	-359 19.9 .013	-160 20 0	0 20 0
.1	-1038 -2.71 0	-195 7.3 -.36	-72 5.0 -.37	-26.0 3.6 -.31	0 2.7 -.27
.2	-589 -.37 0	-71.7 3.3 -.22	-17.1 1.4 -.16	-4.4 .66 -.10	0 .37 -.07
.4	-300 -.007 0	-22.6 1.0 -.08	-2.3 .23 -.03	-.16 .03 -.007	0 .007 -.003
1.0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0

$t_1 = 40$

V \ W	0	.05	.1	.2	1.0
0	-2400 -20 0	-722 18.1 .14	-359 19.9 .013	-160 20 0	0 20 0
.1	-1684 -5.27 0	-357 13.1 -.62	-138 9.6 -.70	-51 7.0 -.61	0 5.3 -.53
.2	-1358 -1.39 0	-215 9.5 -.62	-61 5.0 -.57	-16.6 2.5 -.39	0 1.4 -.28
.4	-1098 -.097 0	-125 6.1 -.46	-20.3 1.9 -.28	-2.2 .38 -.09	0 .10 -.04
1.0	-800 0 0	-65 3.2 -.24	-5.7 .57 -.09	-.05 .01 -.003	0 0 0

[1] Entries in each cell are $\delta x(t_f)$, $\delta \dot{x}(t_f)$, and $\delta \ddot{x}(t_f)$.

Table 2. Terminal Biases for 2° Slope[1]

	W=0	W=.05	W=.1	W=.2	W=1.0
$t_f=120$ (Figure 12)	-5178 0 0	-1294 64.7 -1.16	-619 61.9 -1.90	-279.5 55.9 -2.00	0 49.2 -2.06
$t_f=60$ (Figure 13)	-2060 0 0	-1101 55.1 -.04	-573 57.3 -1.216	-278.3 55.7 -1.93	0 49.2 -2.06
$t_f=120, b_0=0$ (Figure 14)	0 0 0	-1085 54.2 -.38	-607 60.7 -1.71	-279.5 55.9 -2.00	0 49.2 -2.06
Slope bias treated as initial alt. error (Figure 15)	-5178 0 0	-267 12.9 -.97	-12.9 1.3 -.19	-.03 .006 -.001	0 0 0

Variable Duration

		$t_1=20$	$t_1=40$	$t_1=60$	$t_1=80$	$t_1=100$
$t_f=120$ (Fig. 16)	W=.1	-4.85 .49 -.07	-16.8 1.68 -.25	-45.7 4.57 -.68	-115.0 11.50 -1.72	-272 27.22 -4.08
	W=1.0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0

		$t_1=10$	$t_1=20$	$t_1=30$	$t_1=40$
$t_f=60$ (Fig. 17)	W=.1	-27.1 2.71 -.41	-68.6 6.86 -1.0	-132 13.2 -2.0	-226 22.6 -3.4
	W=1.0	0 0 0	0 0 0	0 0 0	0 0 0

[1] Entries in each cell are $\delta x(t_f)$, $\delta \dot{x}(t_f)$, $\delta \ddot{x}(t_f)$.

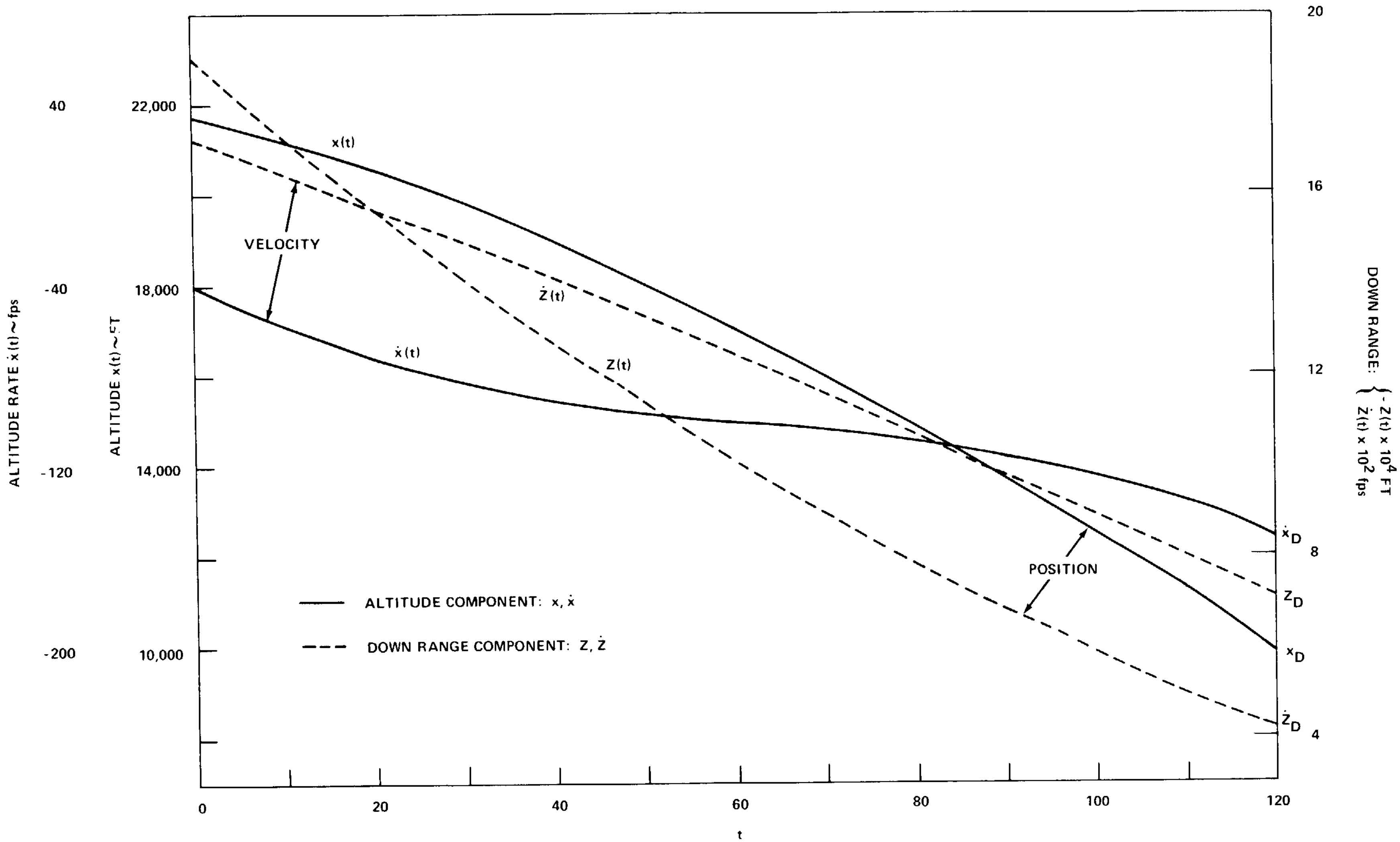

FIGURE 1 - NOMINAL TRAJECTORY BETWEEN THROTTLE-DOWN AND HIGH-GATE - - $x(t)$, $\dot{x}(t)$, $Z(t)$, $\dot{Z}(t)$

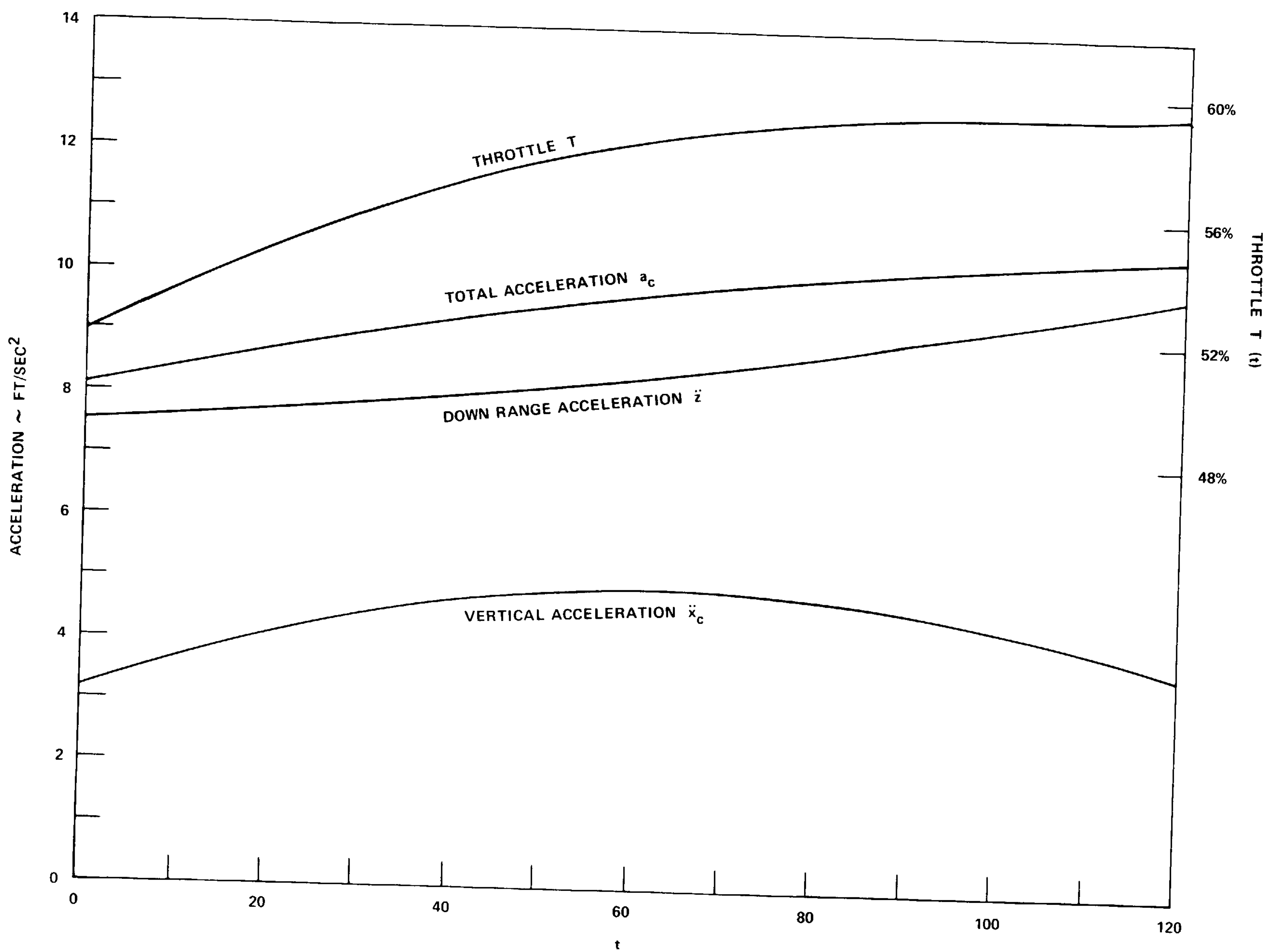

FIGURE 2 - COMMANDED ACCELERATIONS AND THROTTLE FOR NOMINAL TRAJECTORY

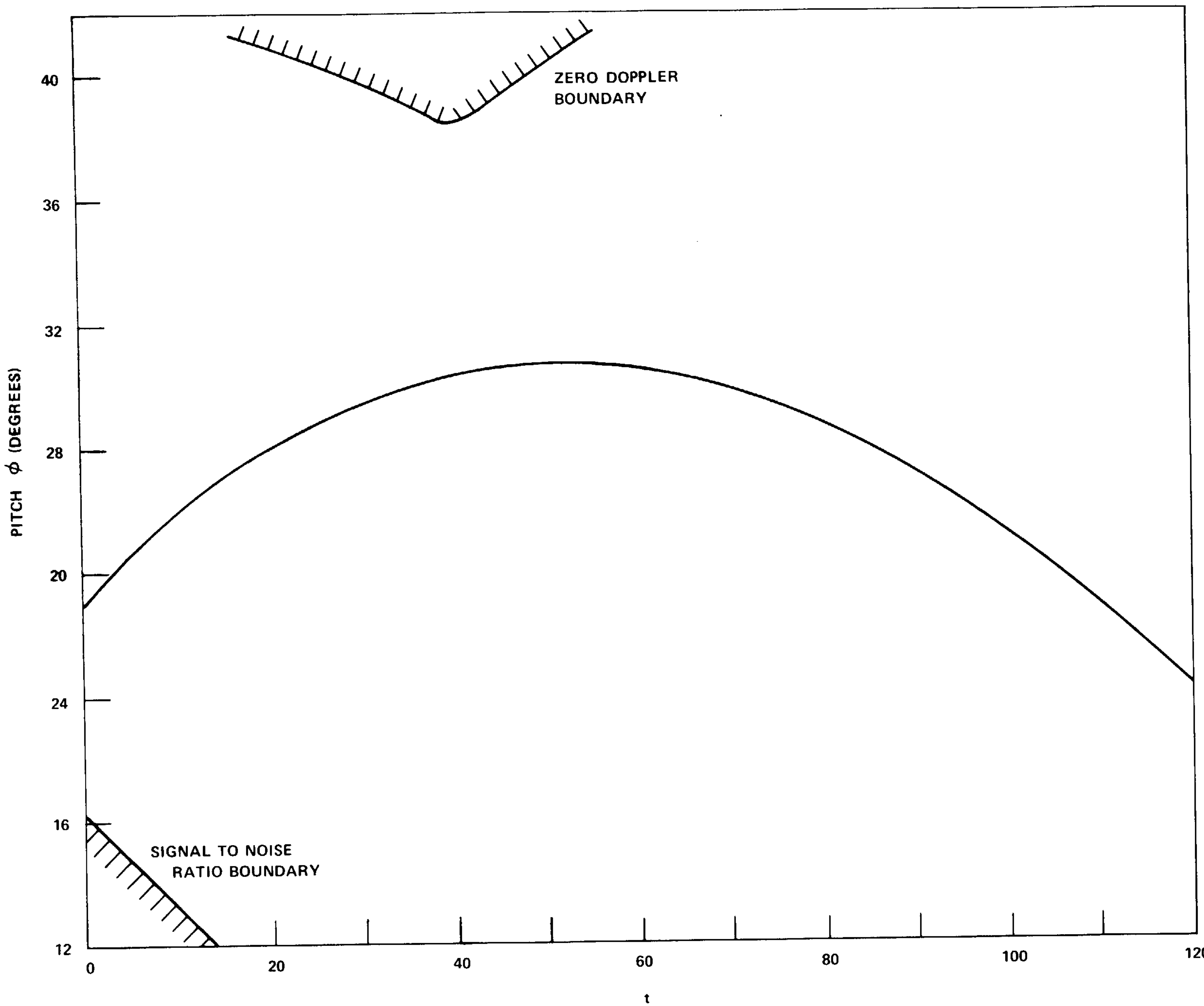

FIGURE 3 - PITCH PROFILE FOR NOMINAL TRAJECTORY

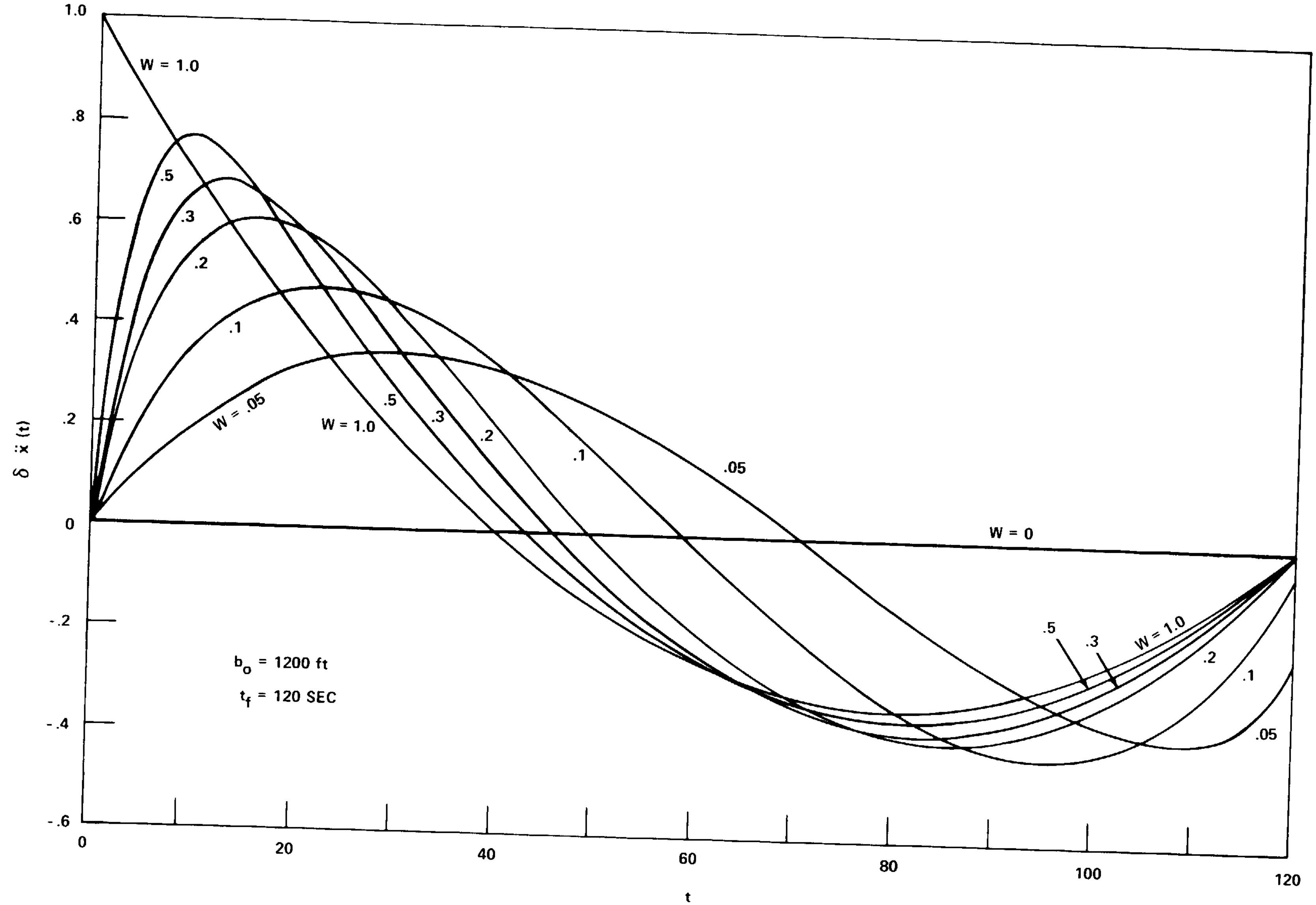

FIGURE 4 - PERTURBED ACCELERATION $\delta \ddot{x}$ DUE TO INITIAL ALTITUDE ERROR b_o

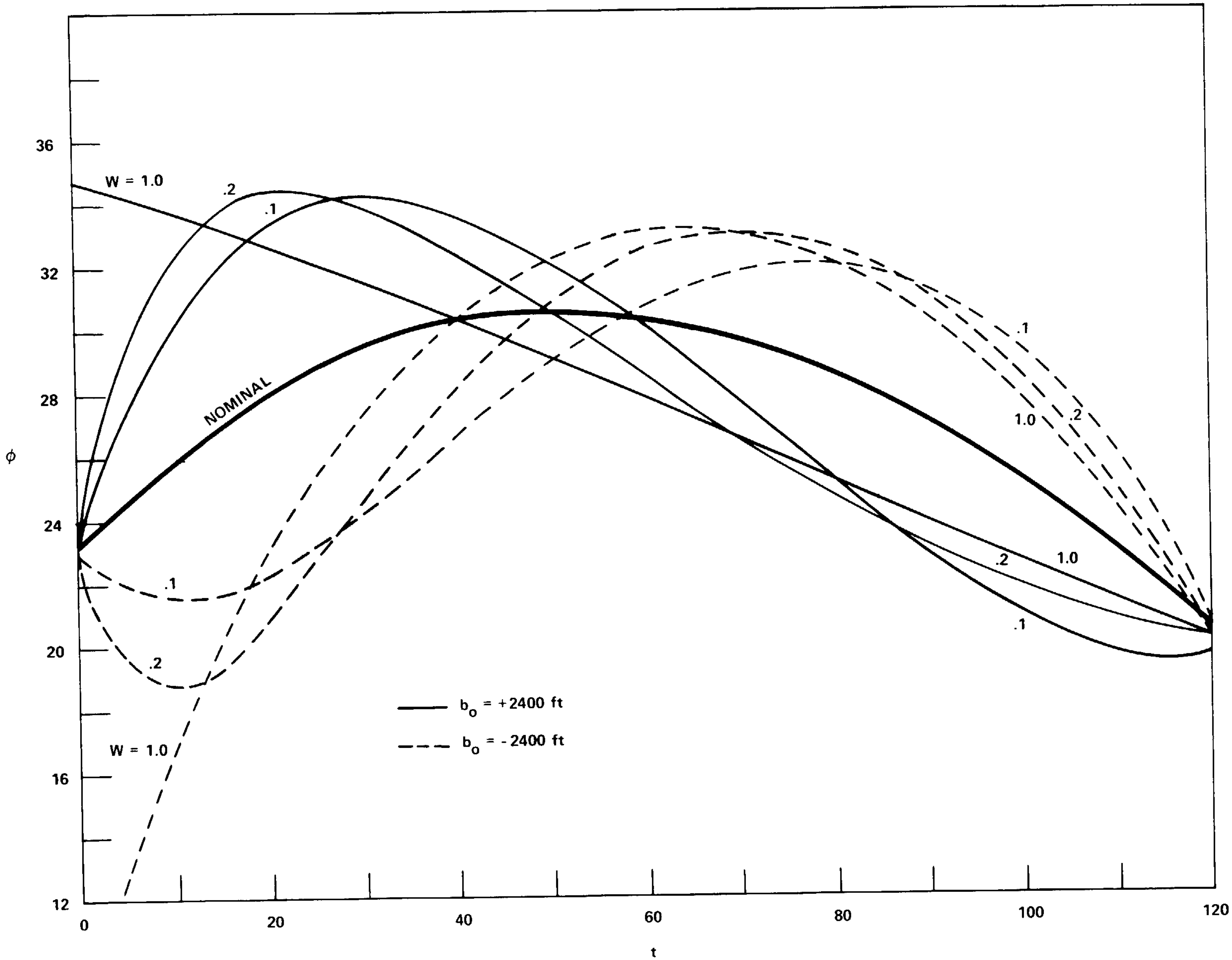

FIGURE 4P - PITCH PROFILES DUE TO INITIAL ALTITUDE ERROR

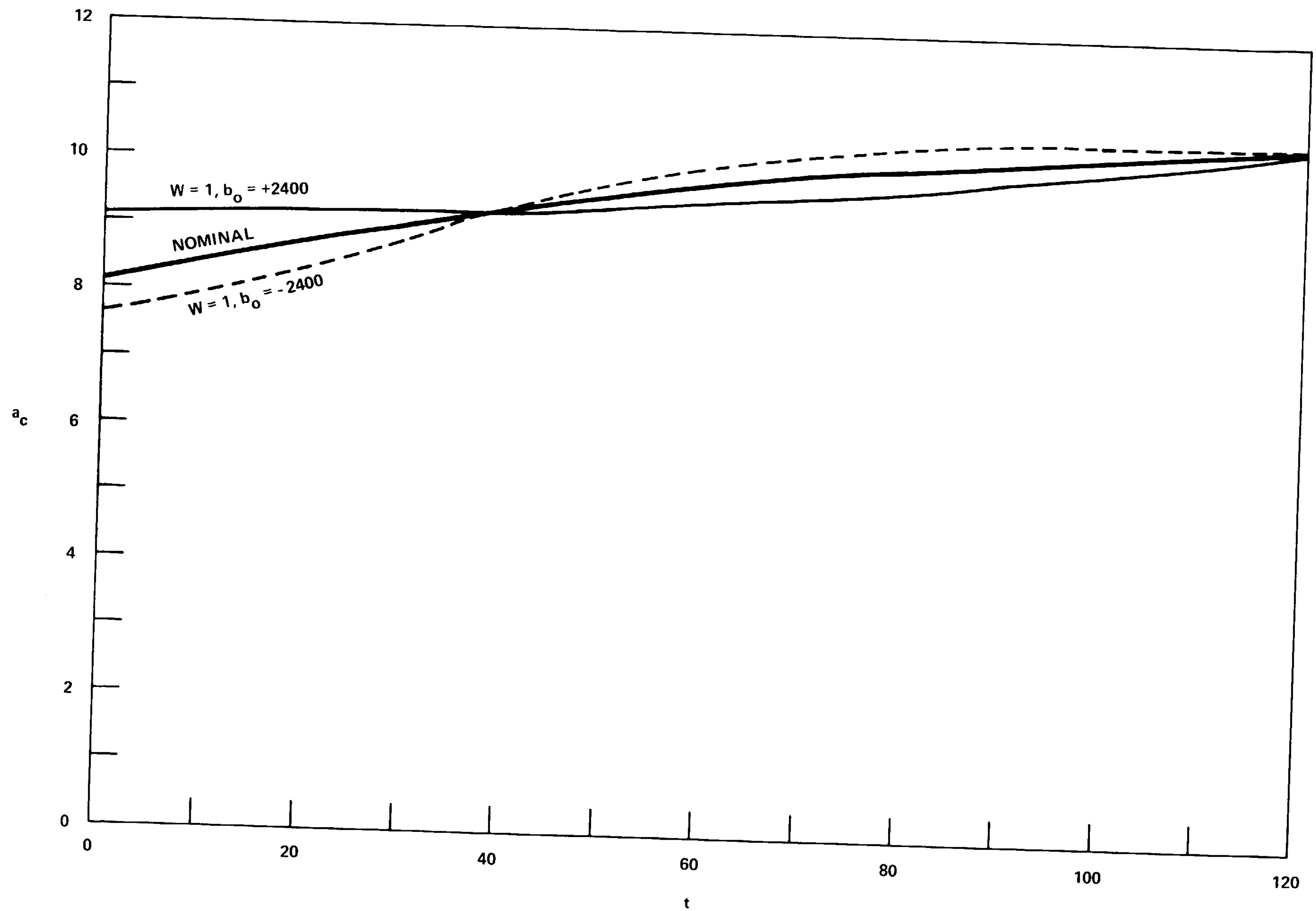

FIGURE 5 - TOTAL COMMANDED ACCELERATION DUE TO INITIAL ALTITUDE ERROR (W = 1)

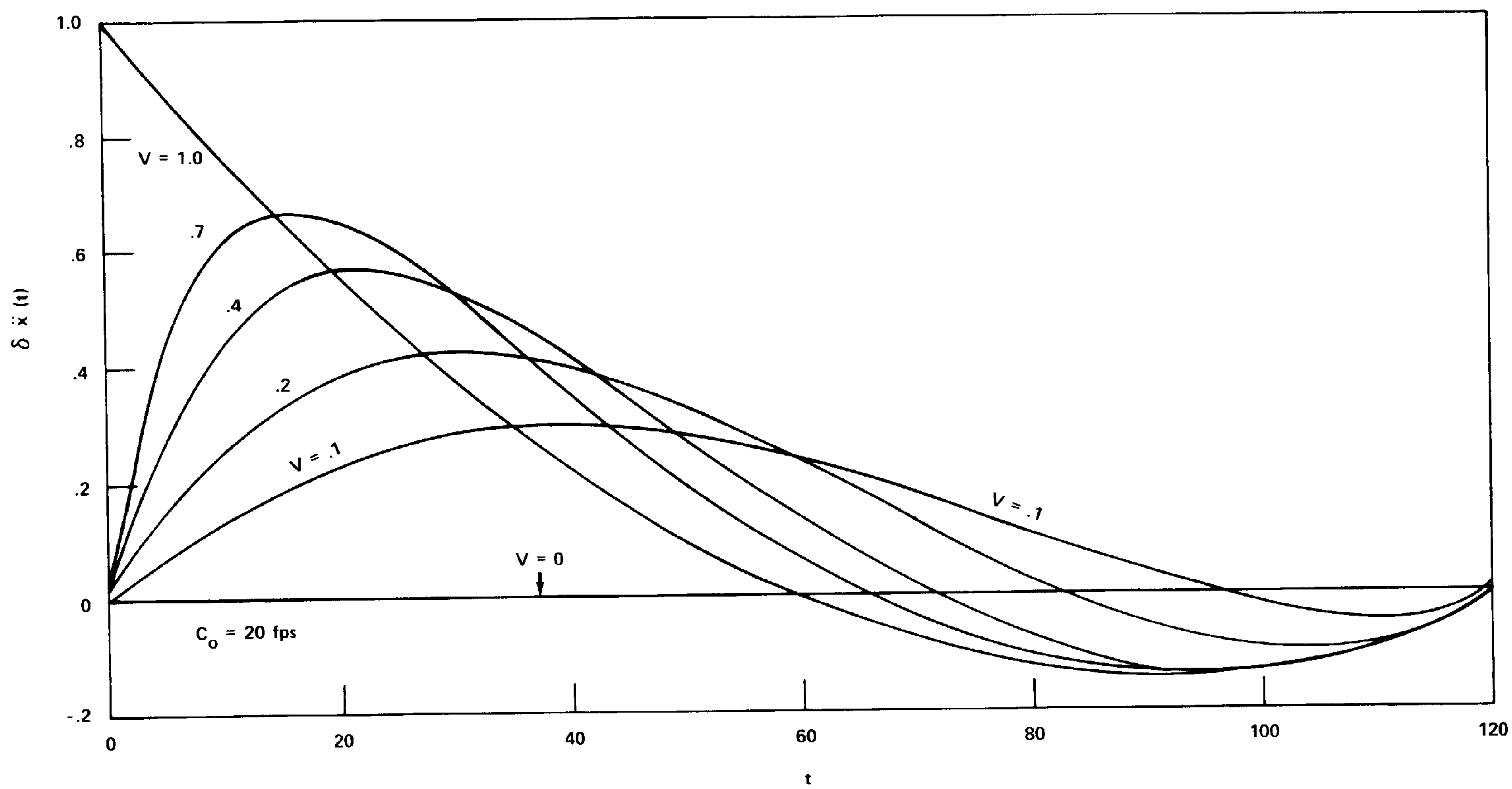

FIGURE 6 - PERTURBATION DUE TO INITIAL VELOCITY ERROR - - VELOCITY FILTERING ONLY (W = 0)

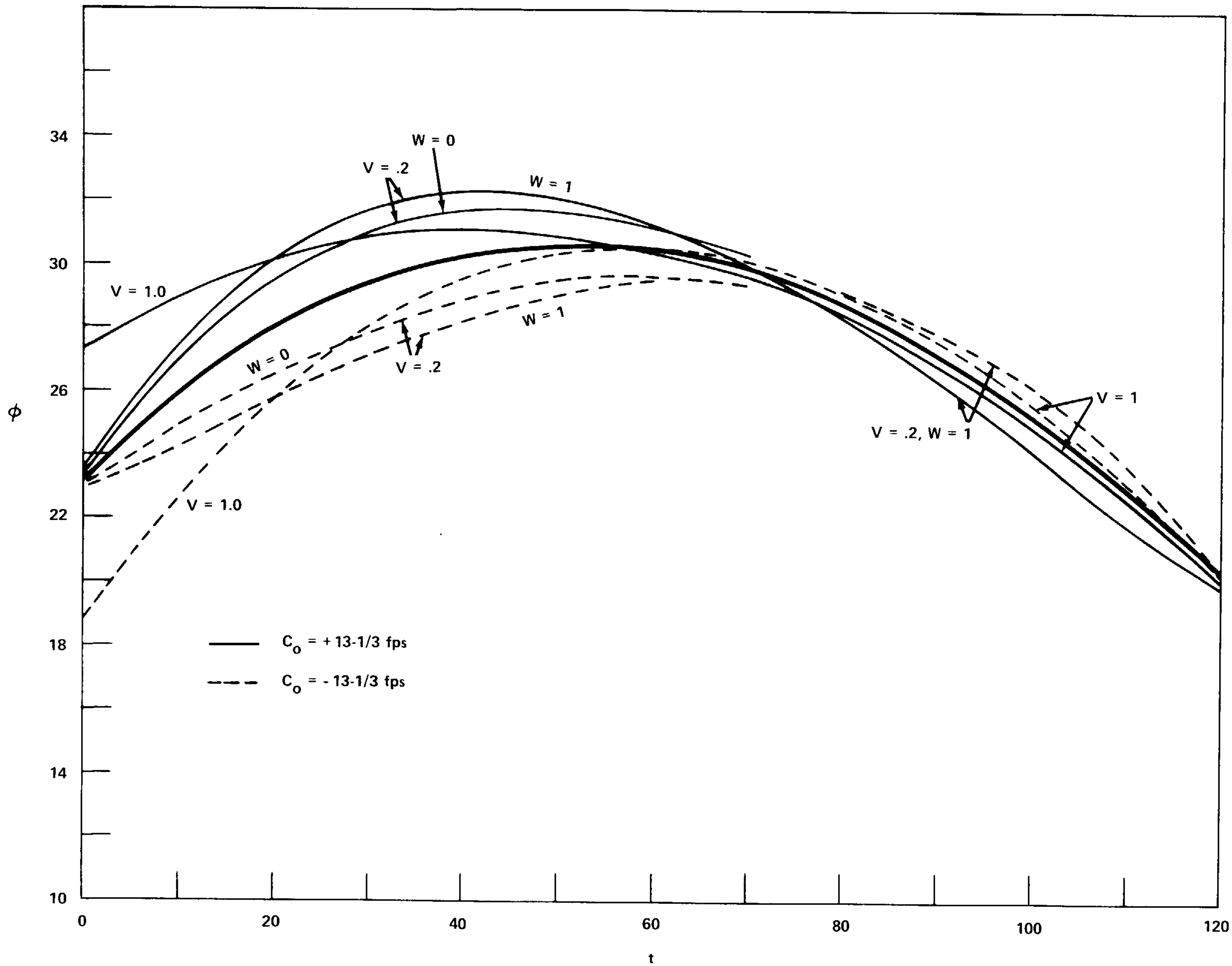

FIGURE 6P - PITCH PROFILES DUE TO INITIAL VELOCITY ERROR - - ALTITUDE FILTER WEIGHTING W = 0 AND 1

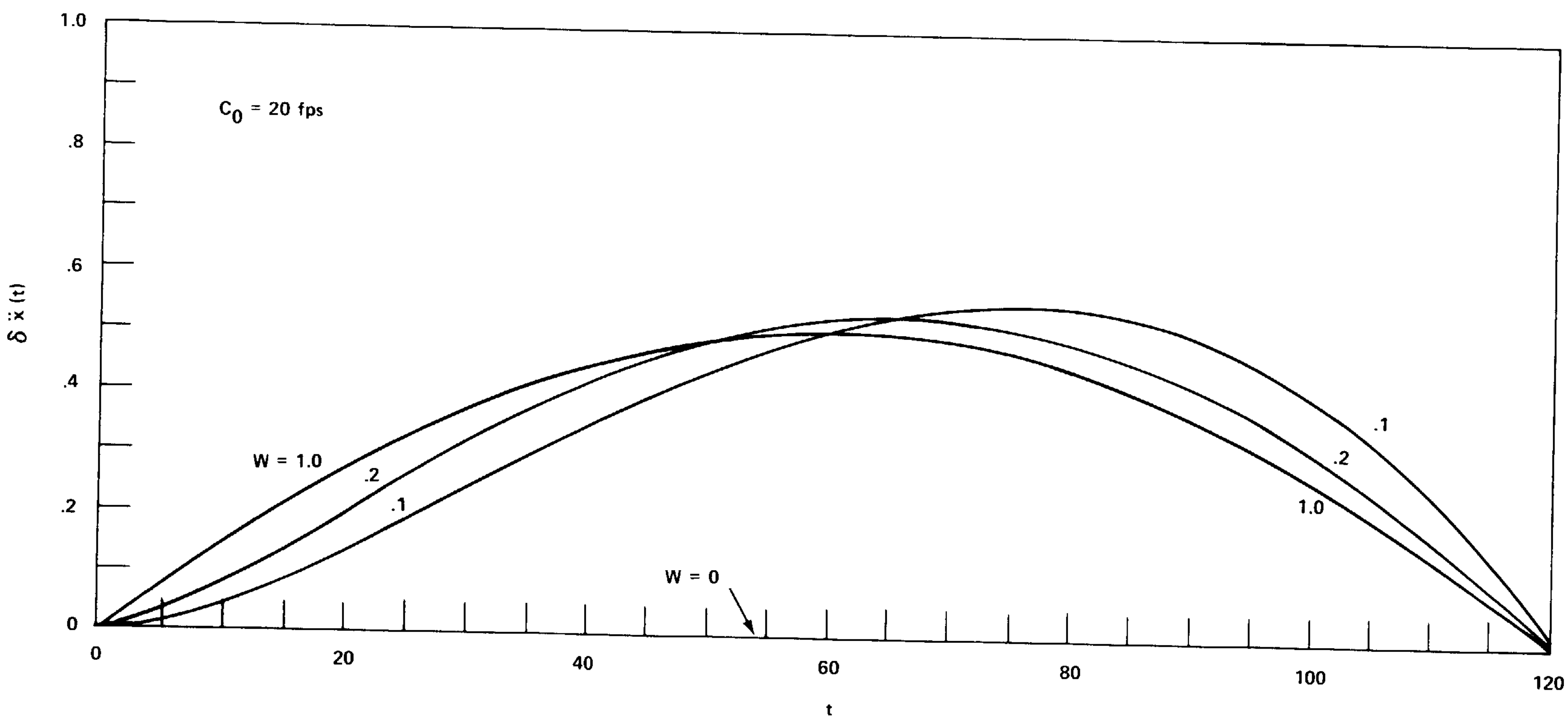

FIGURE 7 - PERTURBATIONS DUE TO INITIAL VELOCITY ERROR - - ALTITUDE FILTERING ONLY (V=0)

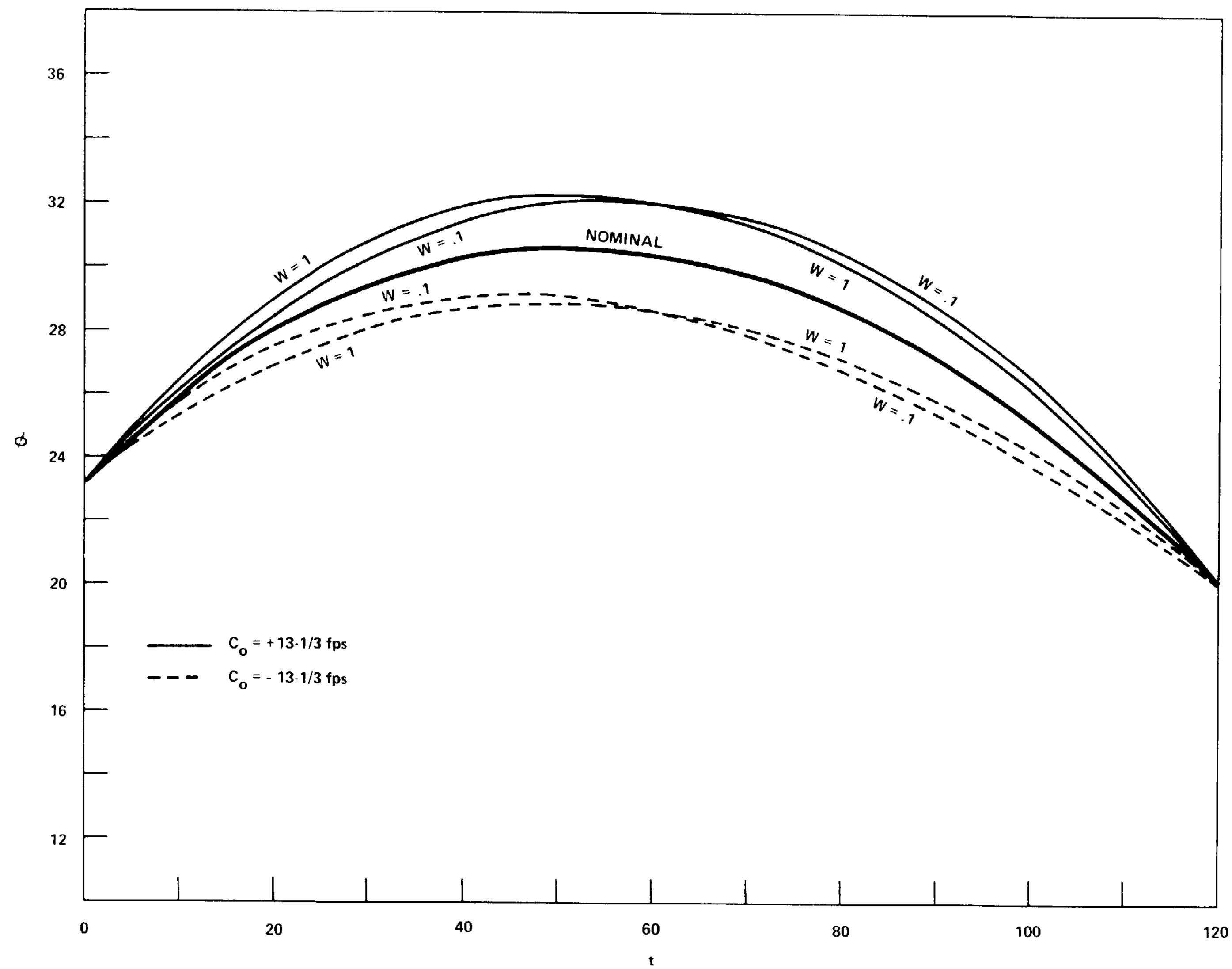

FIGURE 7P - PITCH PROFILES DUE TO INITIAL VELOCITY ERROR - - ALTITUDE FILTERING ONLY (V = 0)

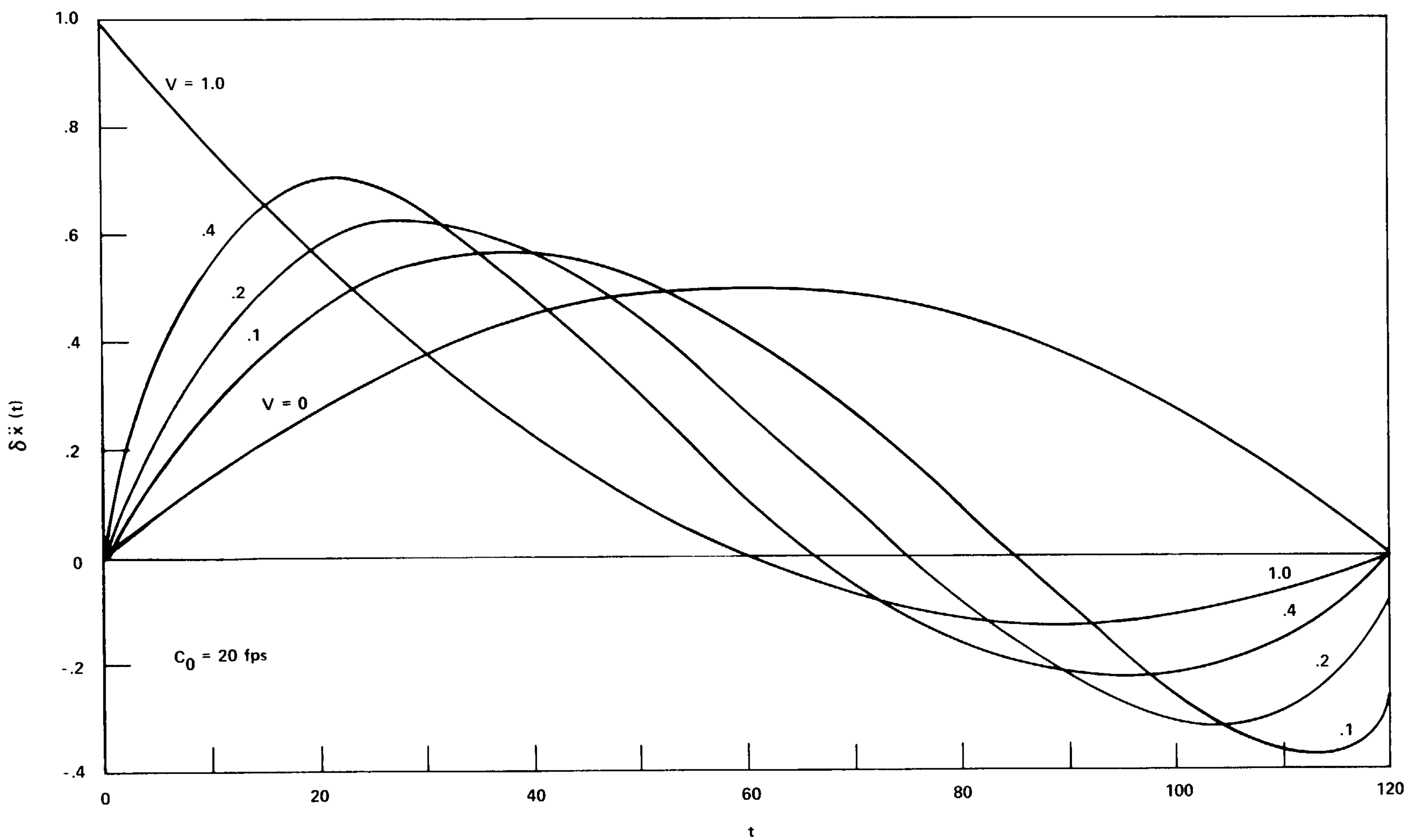

FIGURE 8 - PERTURBATIONS DUE TO INITIAL VELOCITY ERROR - - COMPLETE ALTITUDE CORRECTION (W=1)

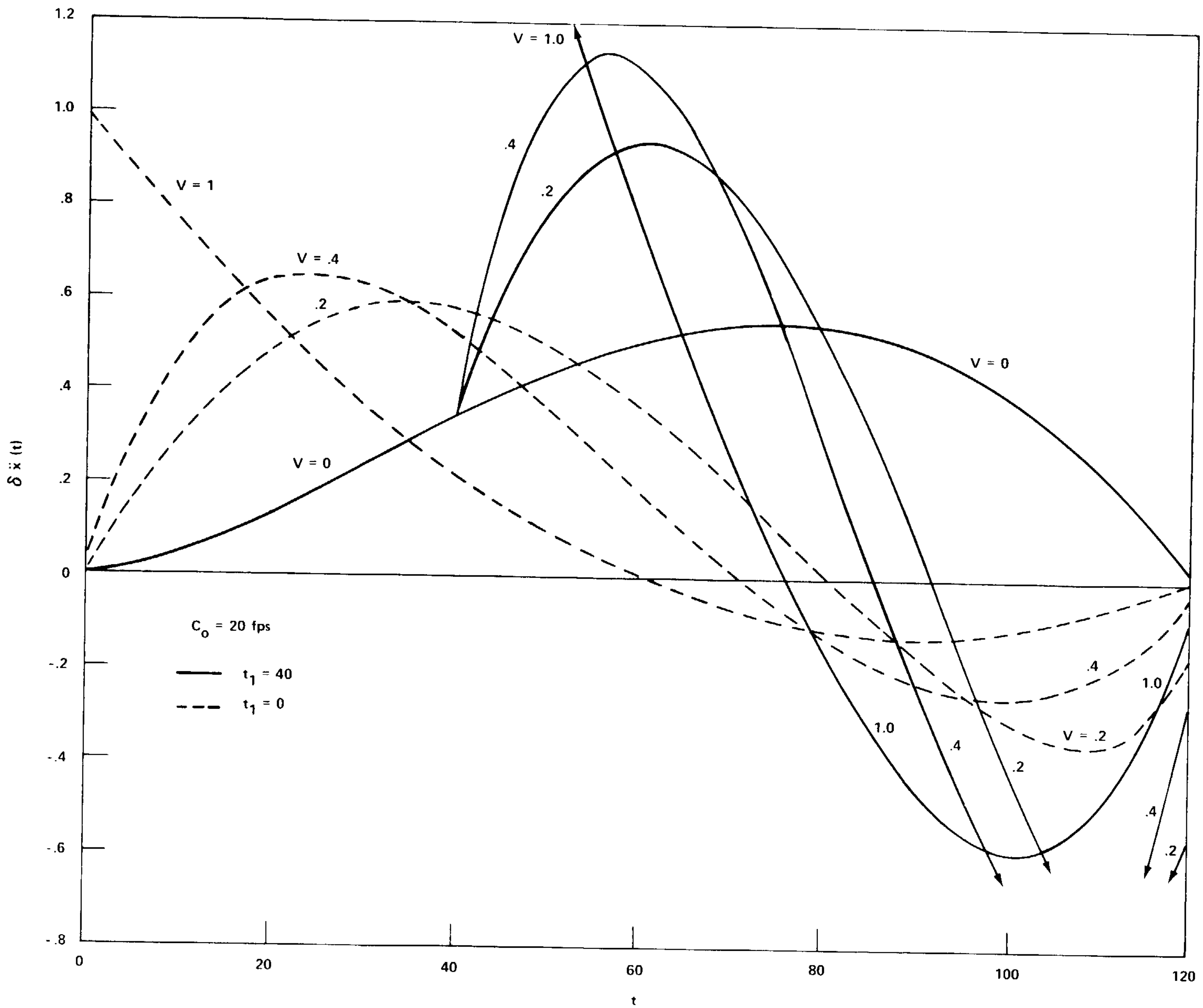

FIGURE 9 - PERTURBATIONS DUE TO INITIAL VELOCITY ERROR - - ALTITUDE FILTER WEIGHTING W = .1 - - t_1 = 0 AND 40 SEC.

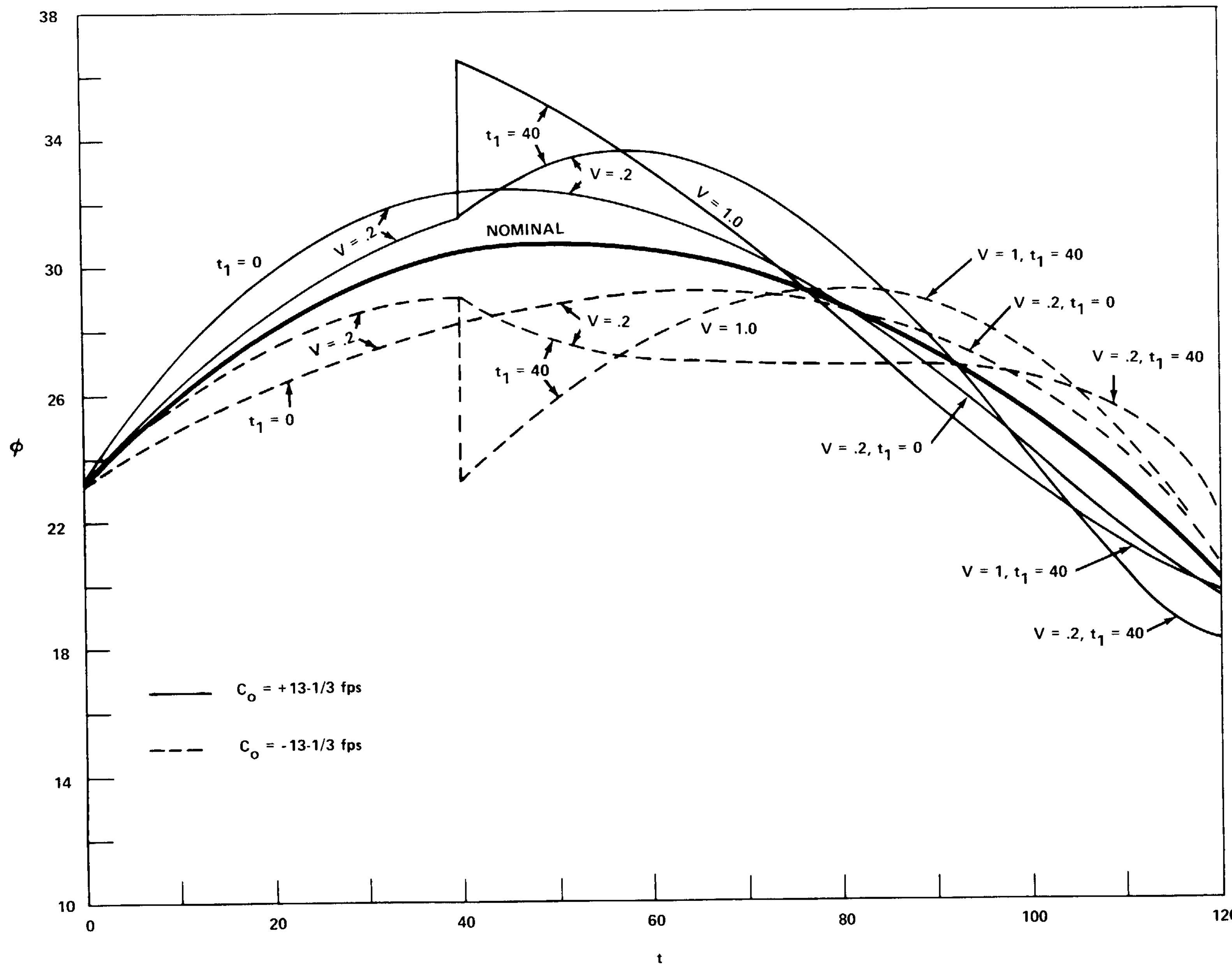

FIGURE 9P - PITCH PROFILES DUE TO INITIAL VELOCITY ERROR - - VELOCITY FILTER STARTING TIME t_1 = 0 AND 40 SEC (W = .1)

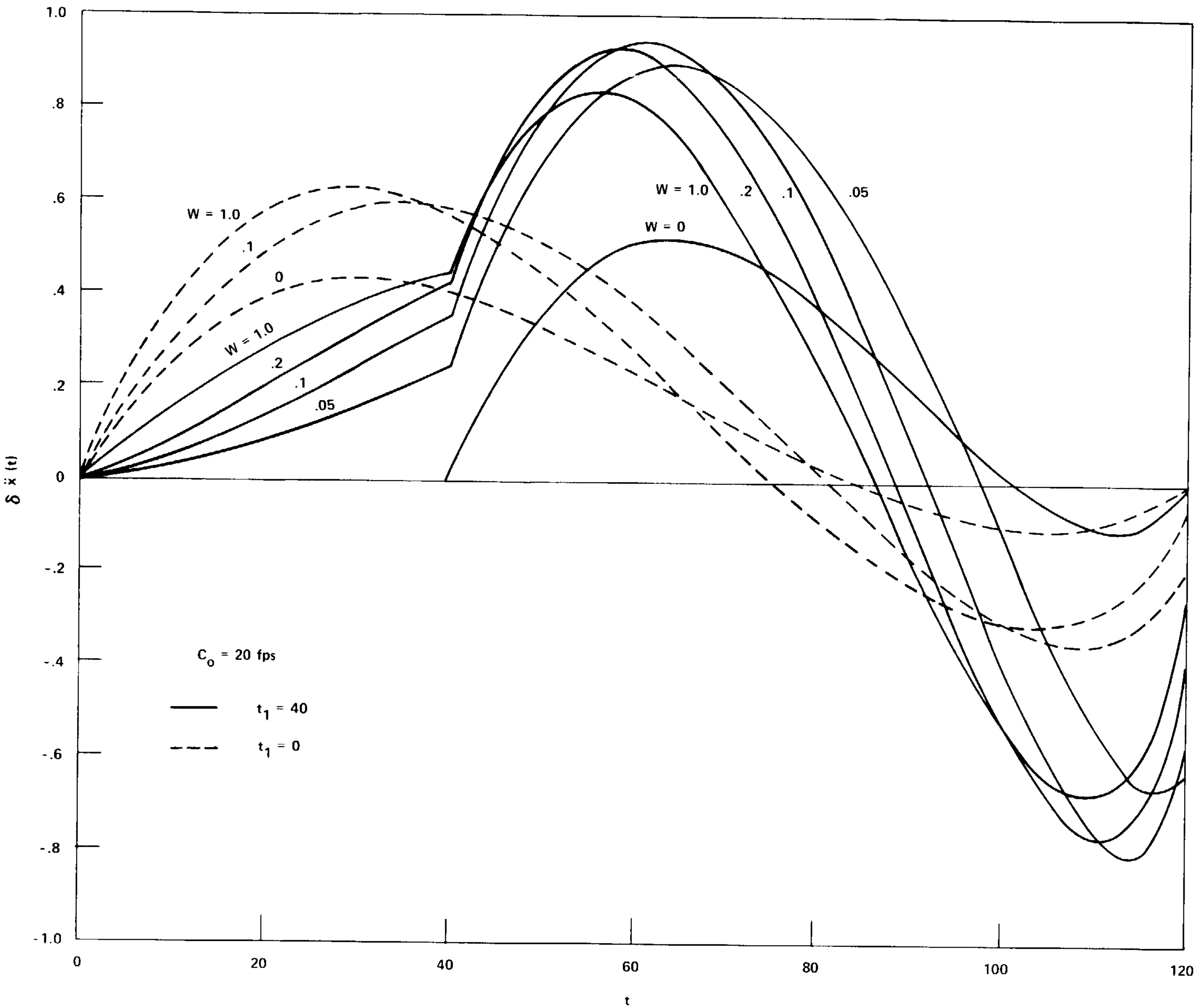

FIGURE 10 - PERTURBATIONS DUE TO INITIAL VELOCITY ERROR - - VELOCITY FILTER WEIGHTING V = .2 STARTING AT t_1 = 0 AND 40 SEC.

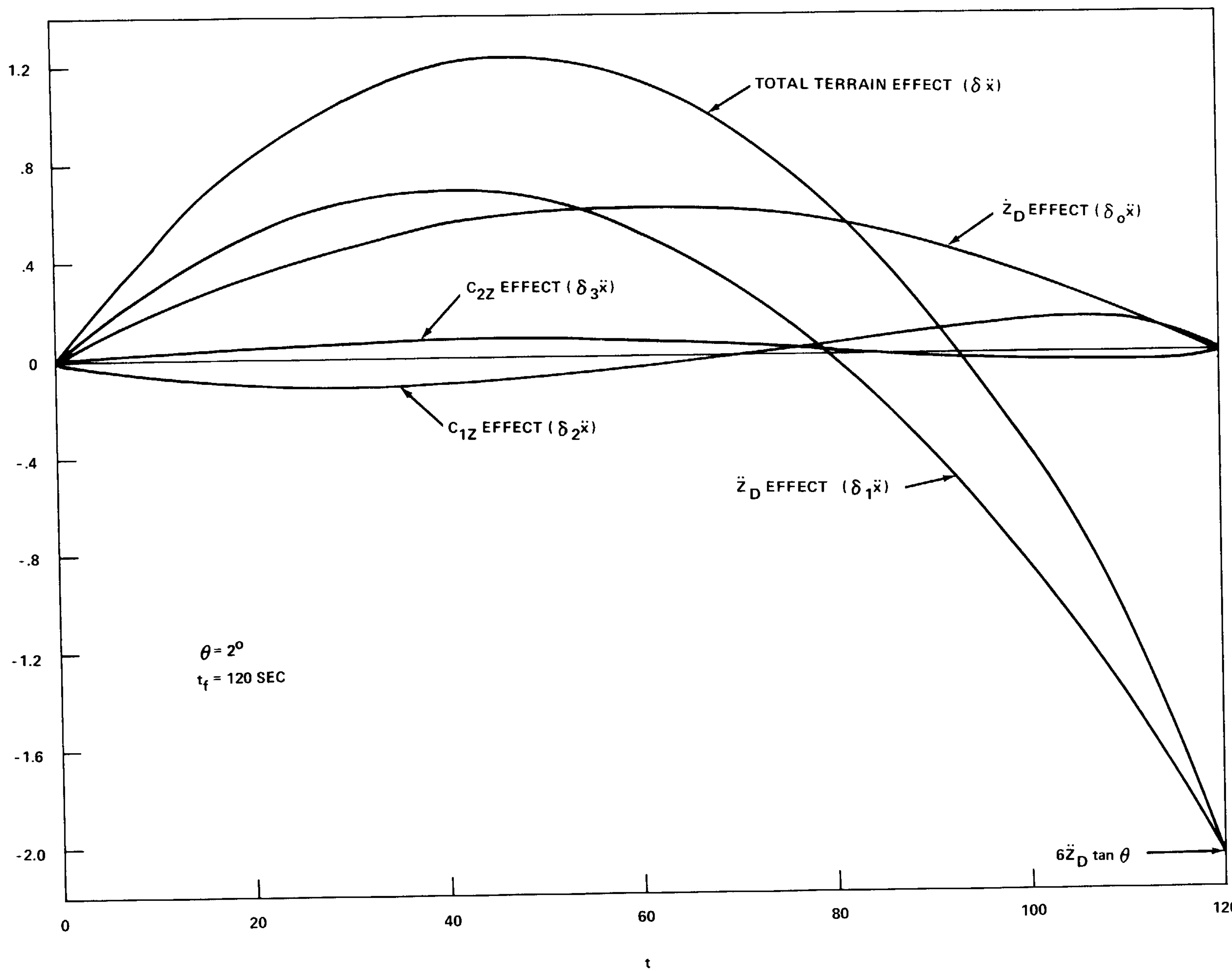

FIGURE 11 · COMPONENTS OF PERTURBATION DUE TO PURE TERRAIN BIAS (W = 1)

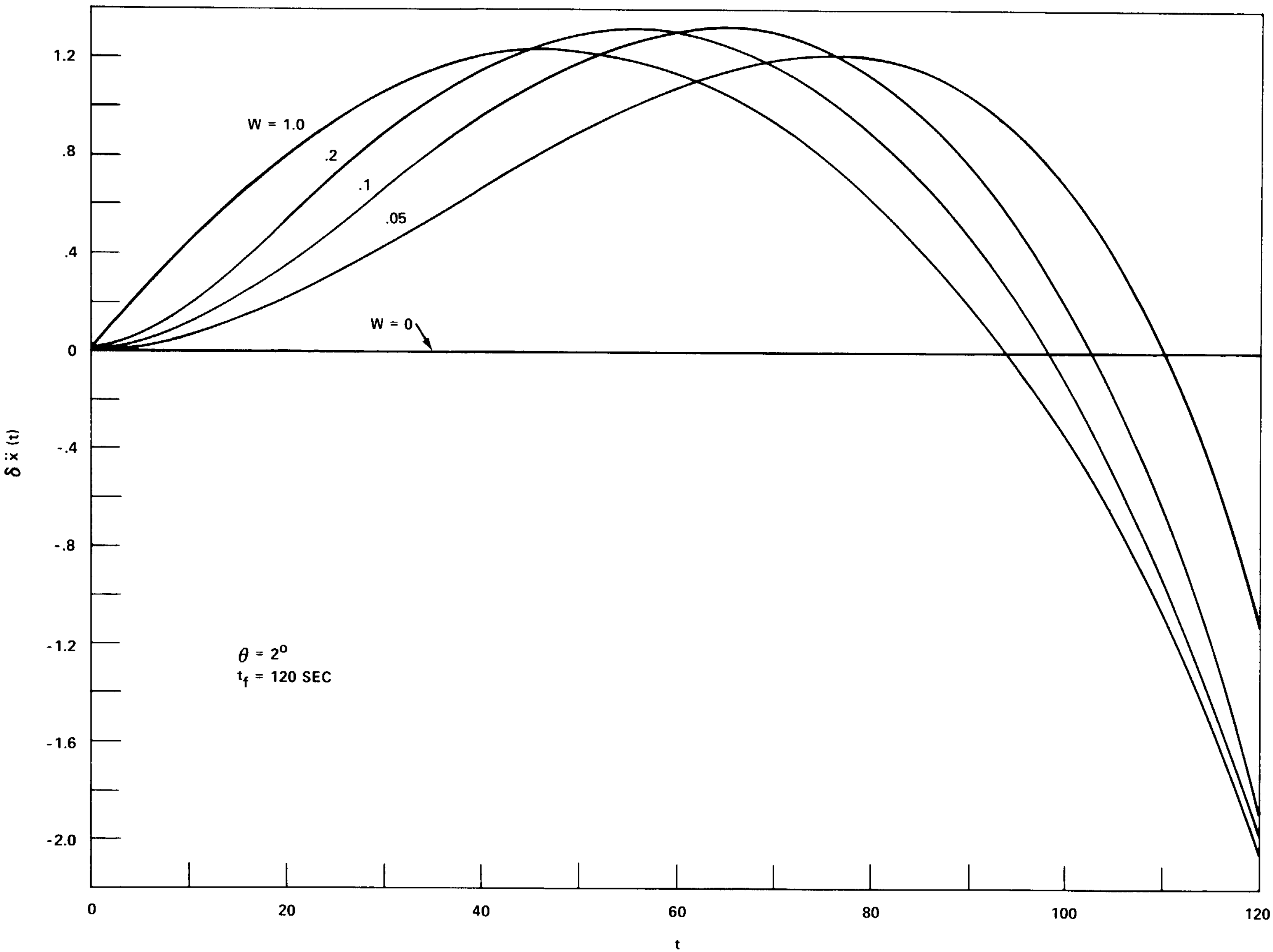

FIGURE 12 - PERTURBATIONS DUE TO CONSTANT SLOPE

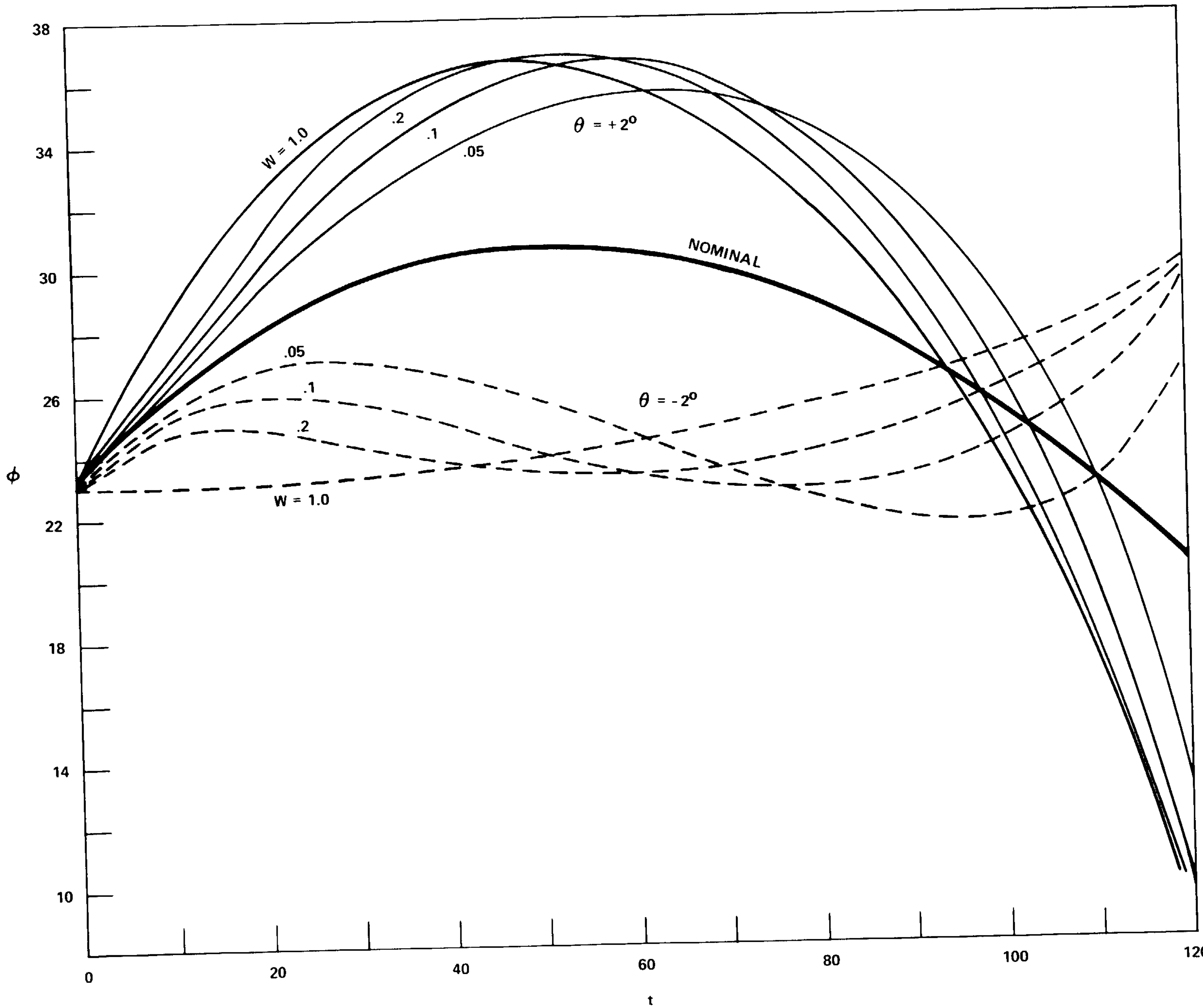

FIGURE 12P - PITCH PROFILES DUE TO CONSTANT SLOPE - - t_f = 120 SEC

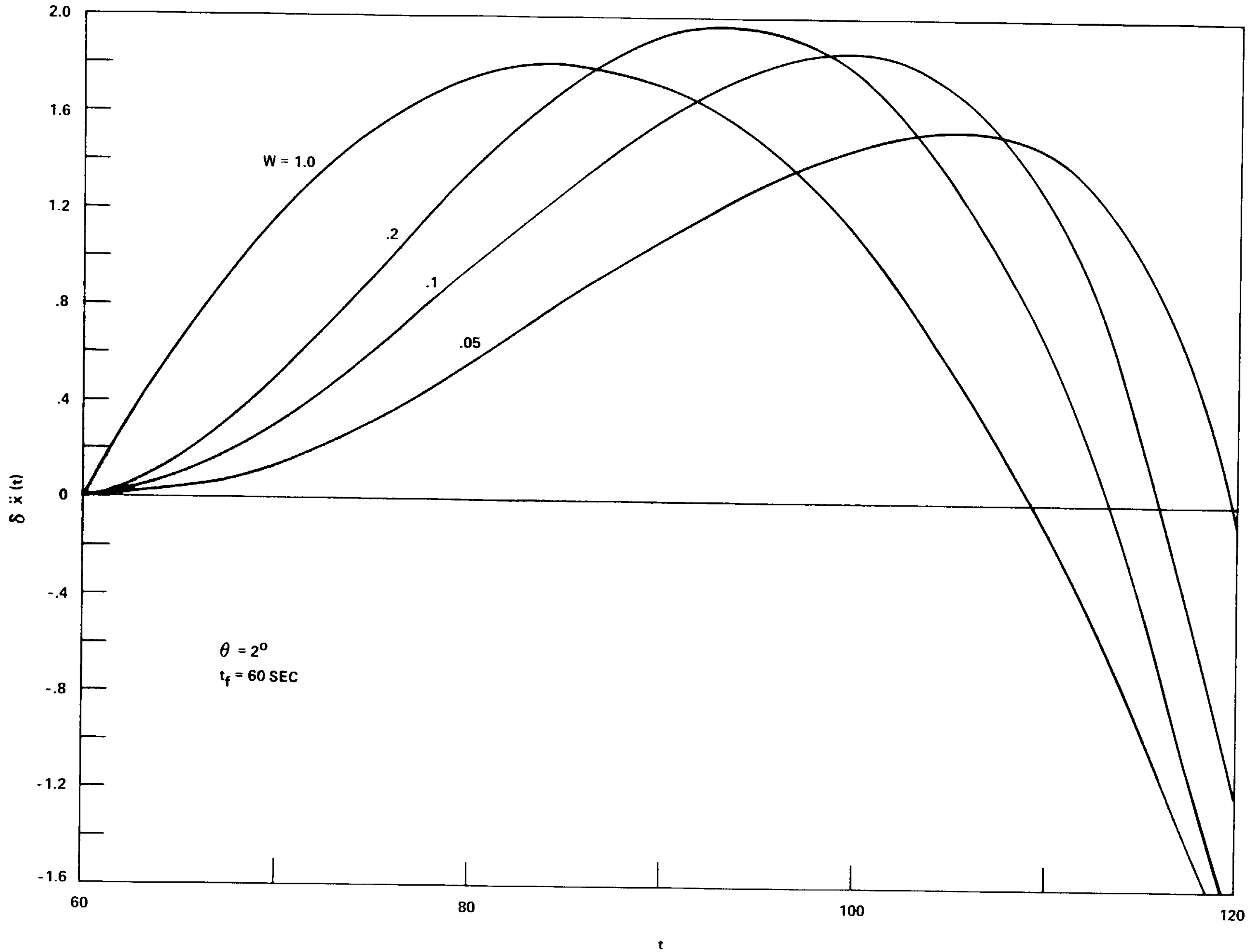

FIGURE 13 - PERTURBATIONS DUE TO CONSTANT SLOPE

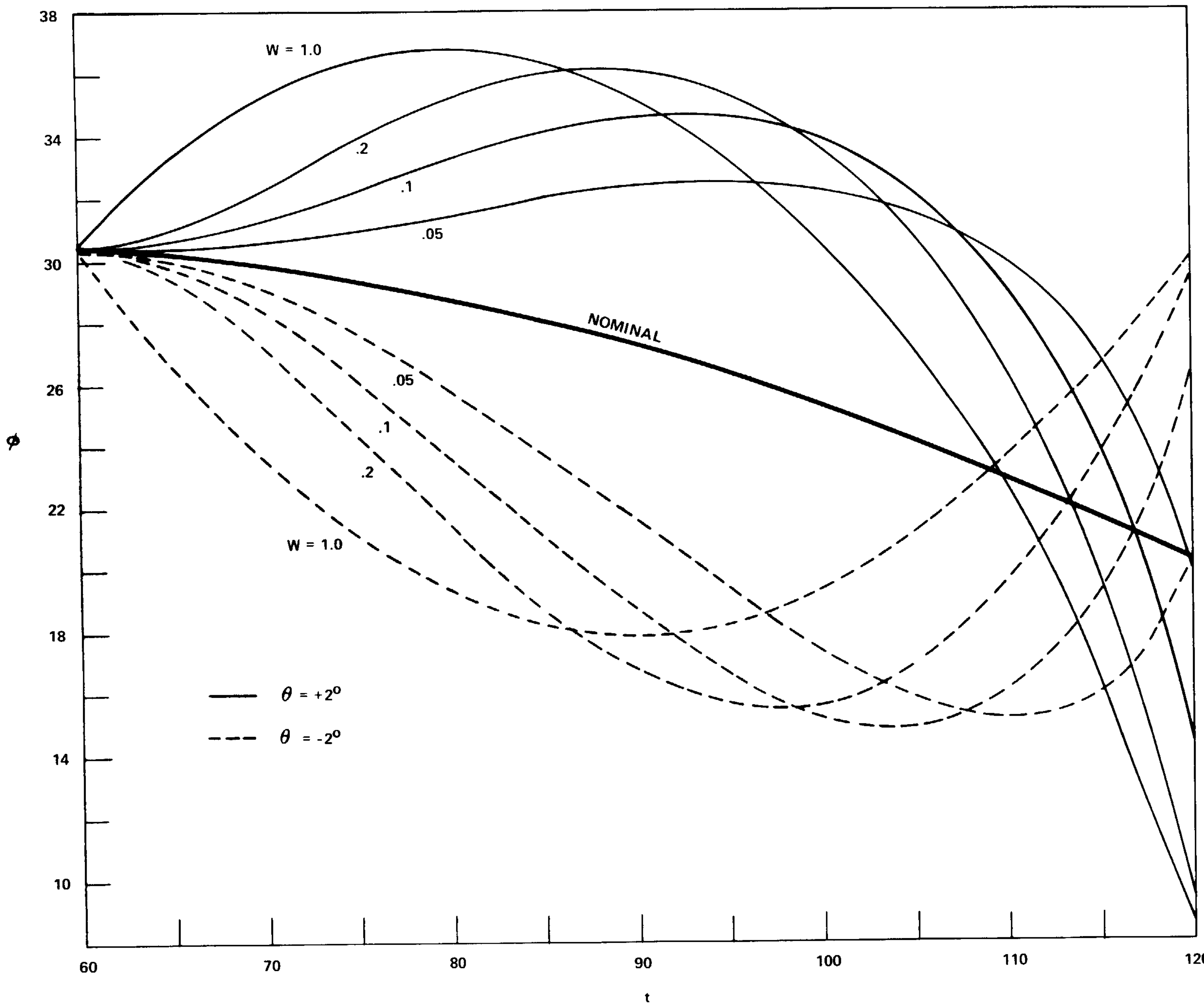

FIGURE 13P - PITCH PROFILES DUE TO CONSTANT SLOPE - - t_f = 60 SEC

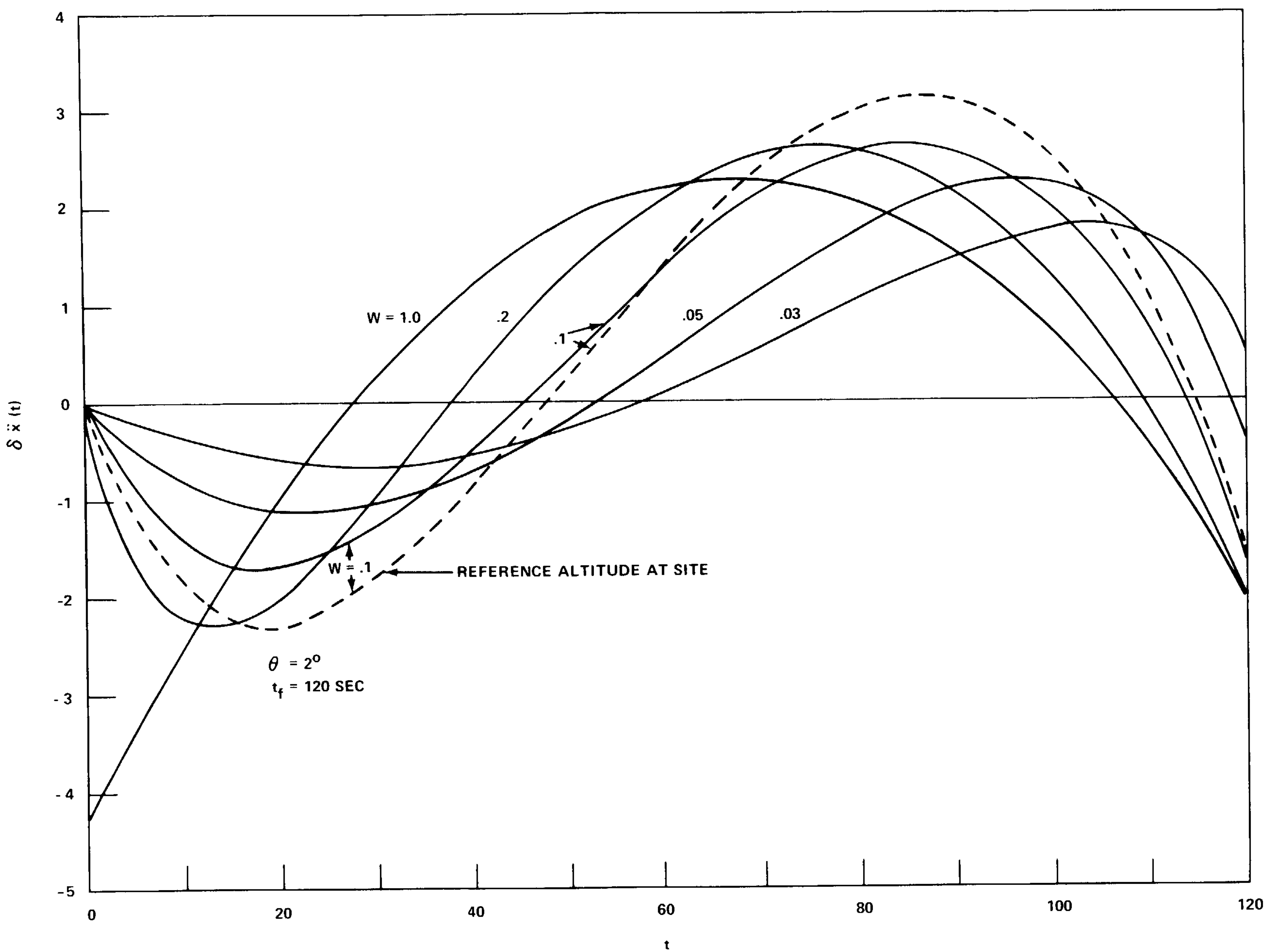

FIGURE 14 - PERTURBATION DUE TO CONSTANT SLOPE WHEN INITIAL IMU BIAS IS ZERO

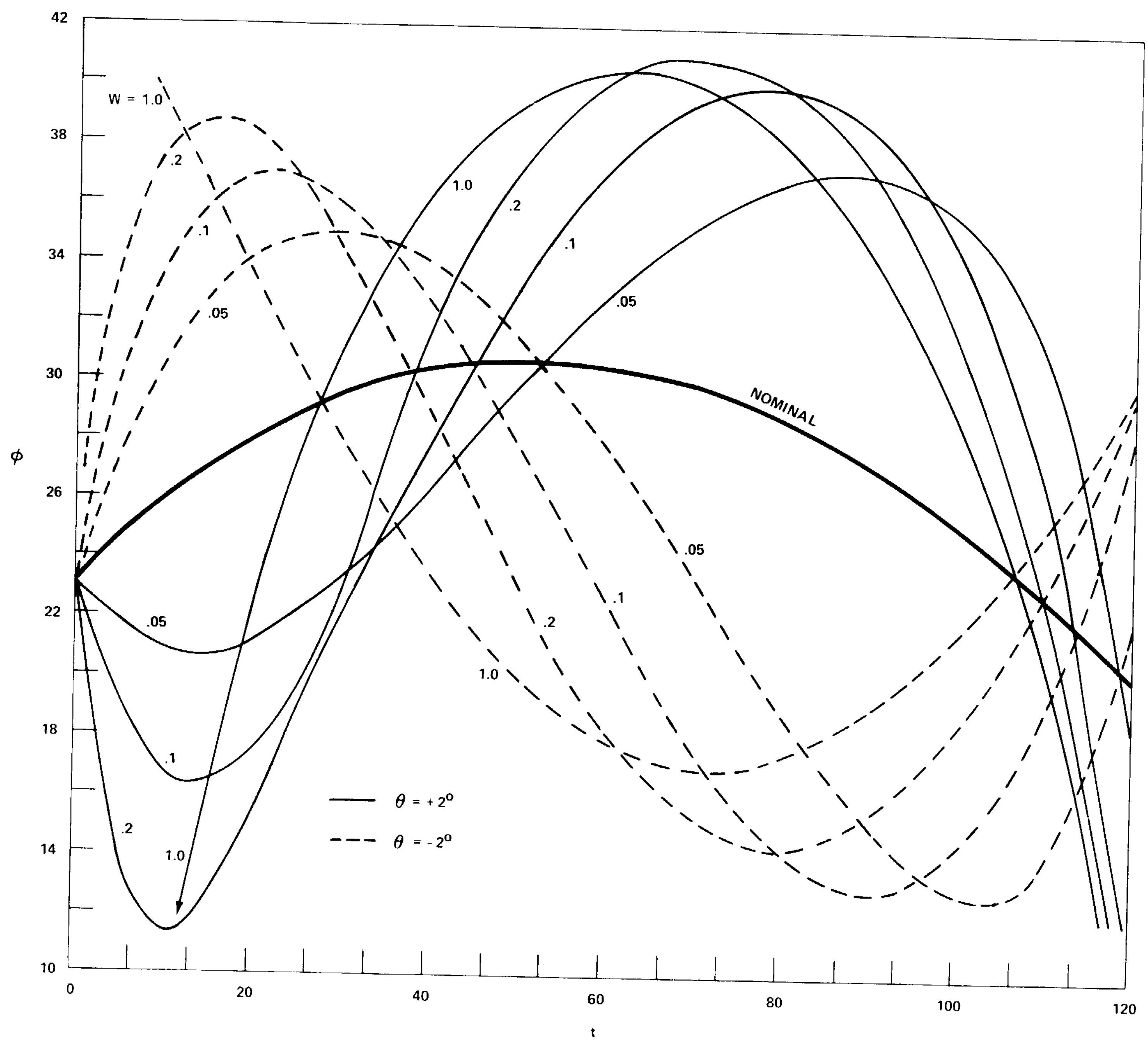

FIGURE 14P - PITCH PROFILES DUE TO CONSTANT SLOPE WHEN INITIAL IMU BIAS IS ZERO

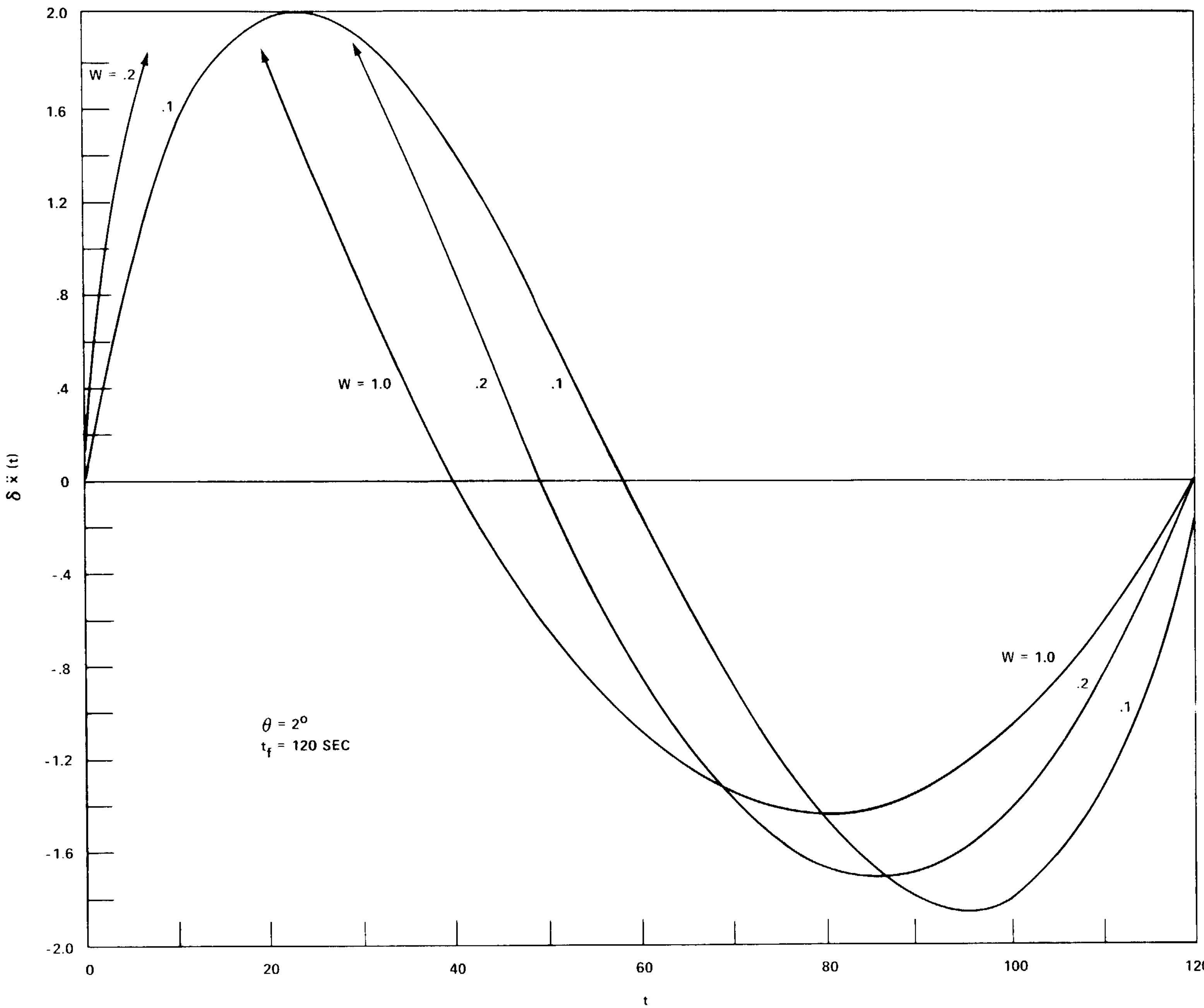

FIGURE 15 - PERTURBATION WHEN SLOPE IS EXTRAPOLATED TO AN INITIAL ALTITUDE ERROR

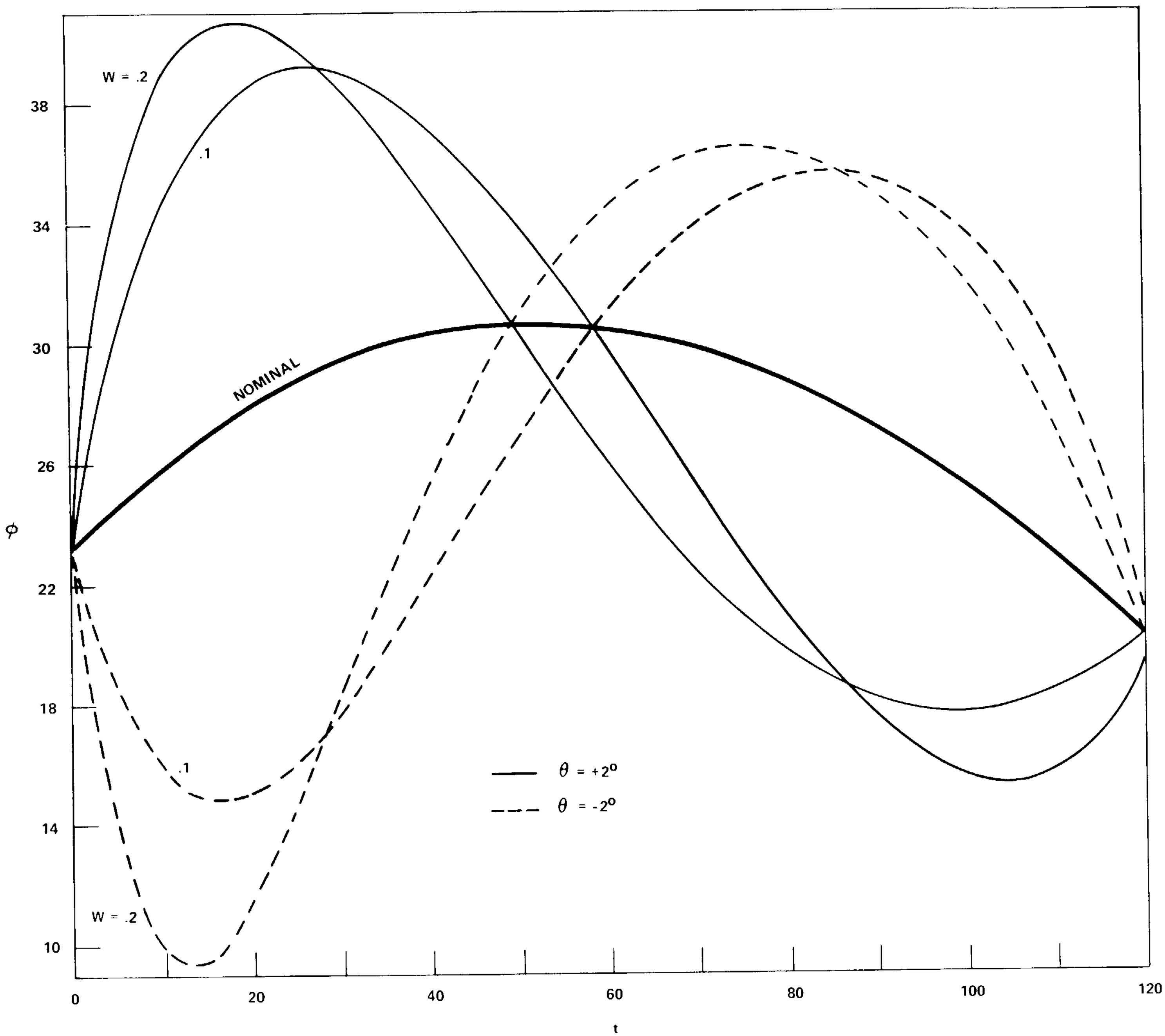

FIGURE 15P - PITCH PROFILES WHEN SLOPE IS EXTRAPOLATED TO AN INITIAL ALTITUDE ERROR

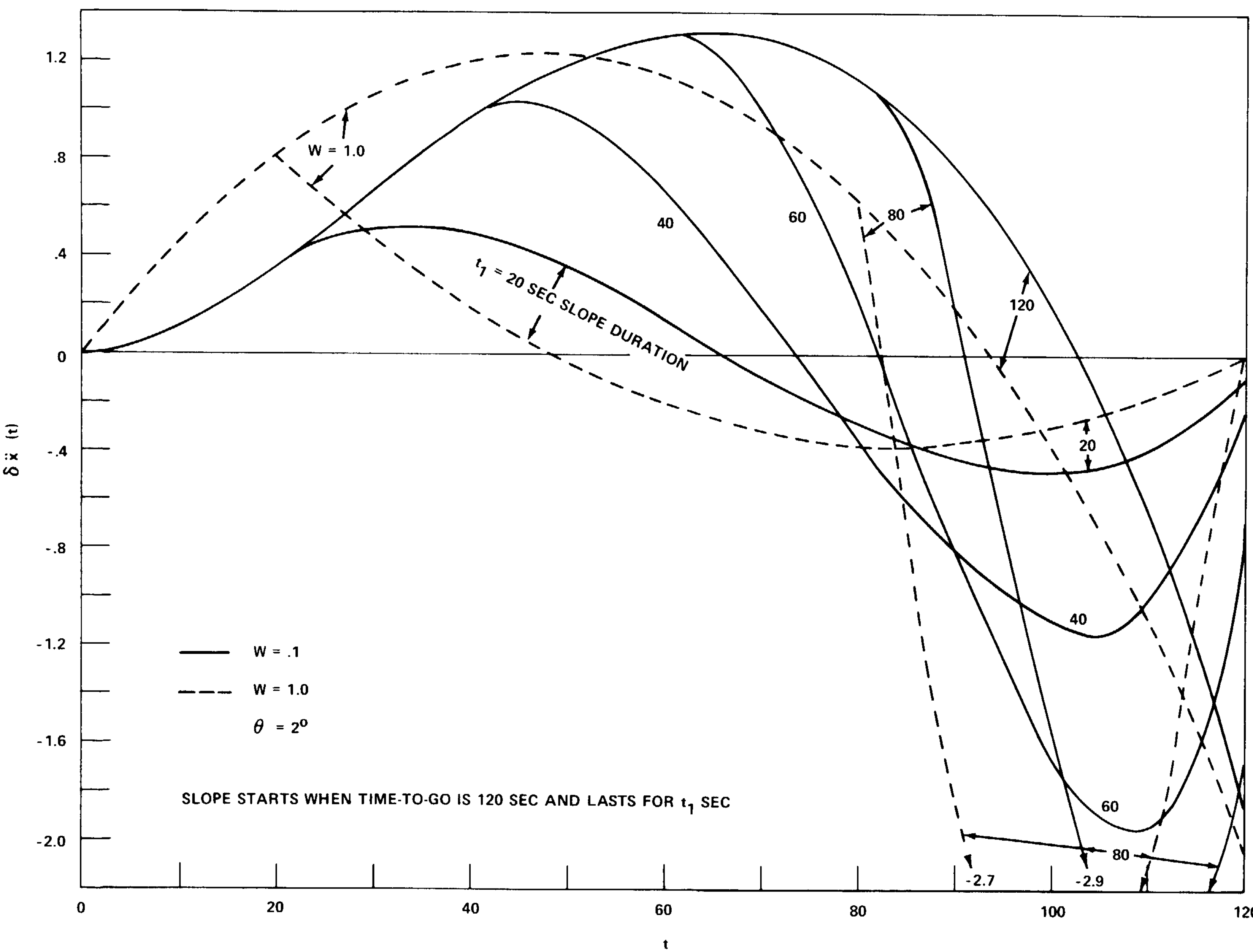

FIGURE 16 - PERTURBATIONS DUE TO SLOPE FOR VARIABLE SLOPE DURATIONS t_1 (t_f = 120)

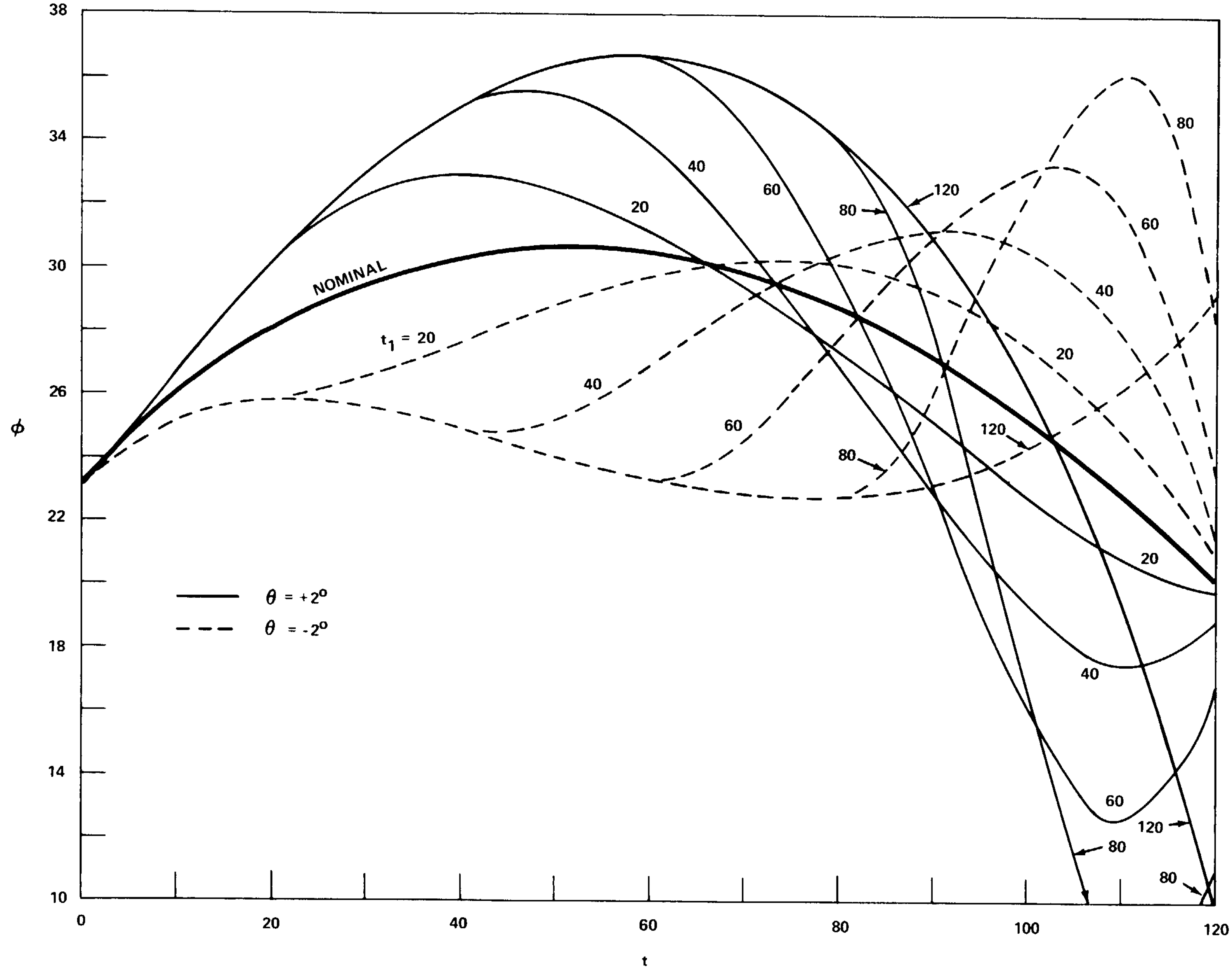

FIGURE 16P - PITCH PROFILES DUE TO CONSTANT SLOPE FOR VARIABLE SLOPE DURATION t_1 (W = .1, t_f = 120 SEC)

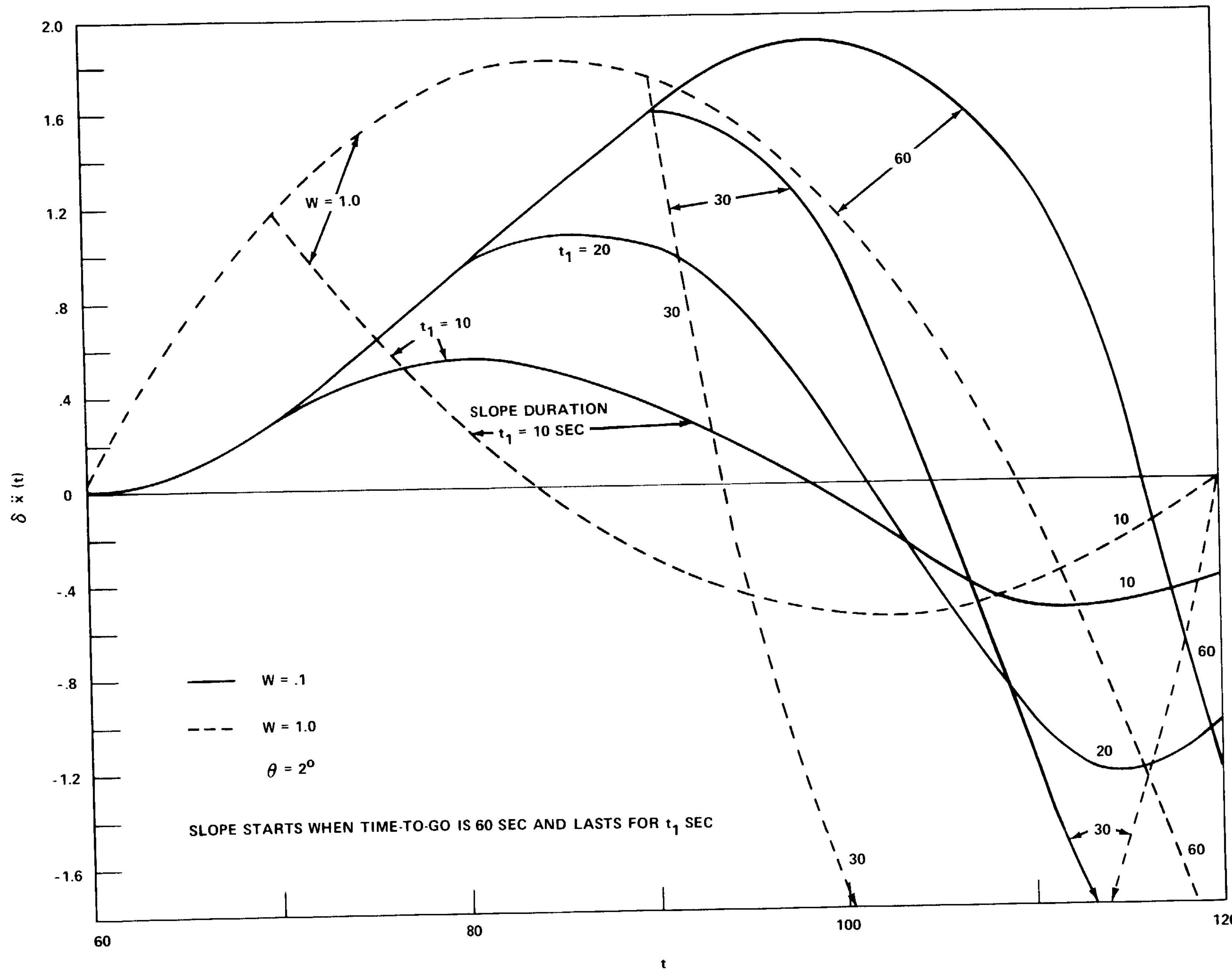

FIGURE 17 - PERTURBATION DUE TO SLOPE FOR VARIABLE SLOPE DURATION t_1 (t_f = 60 SEC)

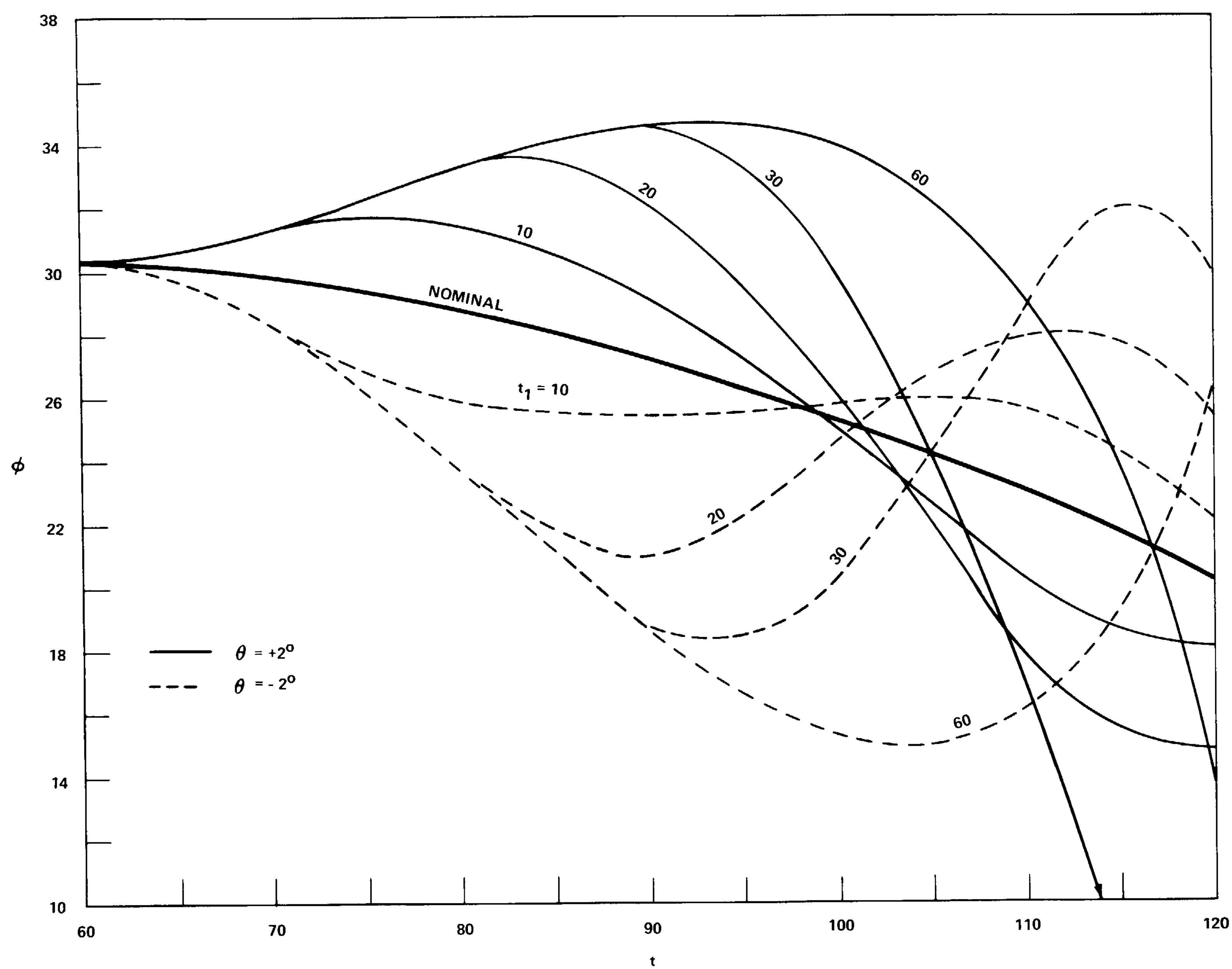

FIGURE 17P - PITCH PROFILES DUE TO CONSTANT SLOPE FOR VARIABLE SLOPE DURATION t_1 (W = .1, t_f = 60 SEC)

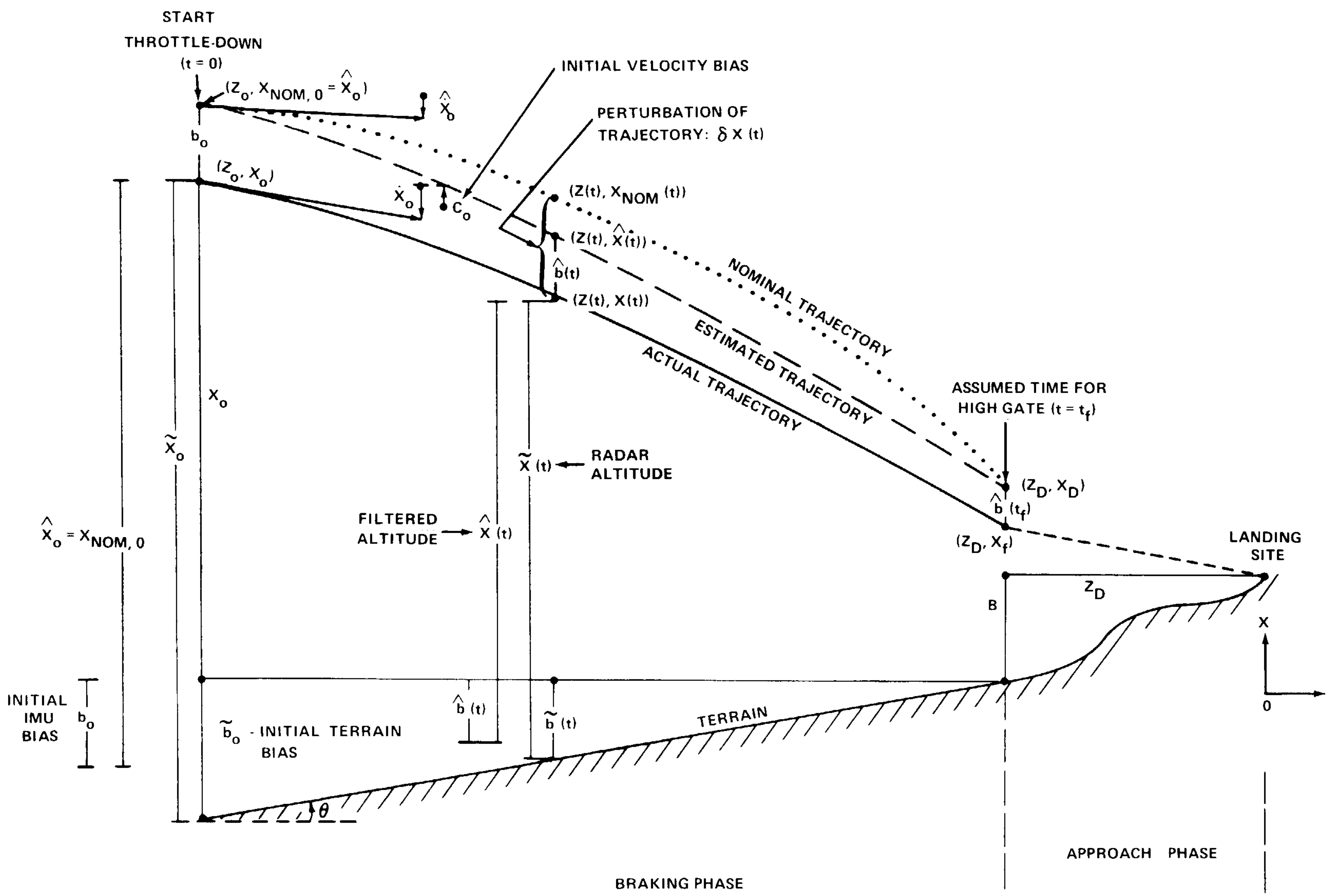

FIGURE A1 - DIAGRAM OF IMU AND TERRAIN BIASES AND PERTURBED TRAJECTORY